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WELL-POSEDNESS OF SINGULARLY PERTURBED NASH GAMES

BENJAMIN FRANKLIN GARDNER

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is presented which transfers fast game information to a modified slow game which leads to a well-posed problem with respect to a modified reduction procedure. For the zero-sum game, it is shown that it does not matter whether the fast modes are due to inadequate or accurate modeling and the usual reduction procedure of singular perturbation leads to a well-posed problem.

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by

Benjamin Franklin Gardner

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WELL-POSEDNESS OF SINGULARLY PERTURBED NASH GAMES

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WELL-POSEDNESS OF SINGULARLY PERTURBED NASH GAMES

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THESIS

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1. INTRODUCTION

1.1. Motivation and Background

In a general multi-input system there may be many decision makers or players each trying to minimize his own performance index. The system is described by a vector differential equation and the performance indices are functions of control input vectors and state vectors over some period of time. We consider the case where the system equation is linear and the performance indices are quadratic functions of state and control vectors. A particular strategy, or rationale for choosing controls, is the Nash equilibrium strategy [1-11] which is appropriate when cooperation among the players cannot be guaranteed.

Definition 1.1: (u_1^*,u_2^*) is called a Nash equilibrium strategy set if

$$J_1(u_1^*,u_2^*) \leq J_1(u_1,u_2^*) \tag{1.1a}$$

$$J_2(u_1^*,u_2^*) \leq J_2(u_1^*,u_2) \tag{1.1b}$$

where u_i is any admissible strategy for player i and J_i is the performance cost to player i.

The Nash equilibrium strategy has the advantage that if one player deviates unilaterally from the Nash strategy his performance cost will not improve. Notice that it is possible that there may be a strategy pair which gives lower performance cost for both players, however that pair will not have the property (1.1). When the sum of the performance indices is zero the game is called zero-sum and the Nash equilibrium set satisfies the familiar saddle-point condition. If the sum of the performance indices is

different than zero the problem is called nonzero-sum. The extension of Definition 1.1 to more than two players is easy.

When the system has slow and fast modes, the control problem is ill-conditioned, that is, it is numerically "stiff." To alleviate this ill-conditioning and to reduce the amount of computation, singular perturbation techniques have been developed, some of which are presented in [12-16]. Typically, the full order singularly perturbed system is written as

$$\dot{x}_1 = f_1(x_1,x_2,u,t,\mu) \tag{1.2a}$$

$$\mu\dot{x}_2 = f_2(x_1,x_2,u,t,\mu) \tag{1.2b}$$

where the states are $x_1 \in R^n$, $x_2 \in R^m$, the control is $u \in R^r$ and μ is the scalar singular perturbation parameter. The idea in regulator systems [16] is to design an approximate control strategy based on lower order slow and fast subsystems. The slow subsystem in the usual perturbation reduction technique is found from

$$\dot{x}_1 = f_1(x_1,x_2,u,t,\mu) \tag{1.3a}$$

$$0 = f_2(x_1,x_2,u,t,\mu). \tag{1.3b}$$

x_2 is solved for from (1.3b) and substituted into (1.3a) to give the slow subsystem. This x_2 is substituted into the performance index for (1.2) and gives a performance index for the reduced problem.

Definition 1.2: We say that a problem, (1.2) and its associated performance index, is well-posed with respect to the usual singular perturbation reduction method if the cost for the full order problem tends to the cost of the reduced system, (1.3) and its associated performance index, as $\mu \to 0^+$.

An analogous definition can be given for the game problem.

Another method to reduce the amount of computation in the Riccati equations is the introduction of a coupling parameter [17]. This idea has been extended to the Nash game problem [18] whereby parameter imbedding and series expansion, approximations of the Riccati gains can be found.

An application of singular perturbation theory to analysis of large scale interconnected linear quadratic systems with fast and slow modes and multiple controllers has been presented in [19]. This paper assumes that each player employs an accurate representation of his area and a "dynamic equivalent" of the rest of the system. Using methods derived from singular perturbation, Pareto optimal strategies are derived which result in first order approximation to the optimal cost functions.

This dissertation examines the extension of singular perturbation control theory to systems with many controllers using Nash strategy but with each controller using the same system representation. We investigate the well-posedness of linear closed-loop Nash strategies with respect to singular perturbation. We restrict ourselves to linear strategies in view of Basar's work on existence of nonlinear Nash strategies [20].

It should be noted that even if we design control strategies which are within $O(\mu)$ to the Nash equilibrium strategies we do not know in what sense they are related to the Nash equilibrium concept. In view of this we have the following definition.

<u>Definition 1.3</u>: A pair $(\bar{u}_1, \bar{u}_2)$ is called an "asymptotic Nash" equilibrium strategy set if

$$J_1(\bar{u}_1,\bar{u}_2) \leq J_1(u_1,\bar{u}_2) + 0(\mu) \tag{1.4a}$$

$$J_2(\bar{u}_1,\bar{u}_2) \leq J_2(\bar{u}_1,u_2) + 0(\mu) \tag{1.4b}$$

where u_1 and u_2 are any admissible strategies.

This indicates that if we have an asymptotic Nash set that unilateral deviation from the asymptotic Nash control will lead to at best an $0(\mu)$ cost improvement. In the limit as $\mu \to 0^+$ the asymptotic Nash pair are a Nash equilibrium pair. In view of Definition 1.3, questions are raised about how the approximate controls found in [18] by parameter imbedding satisfy the Nash solution concept. It is expected that some similar condition like (1.4) could be found for this case.

1.2. Chapter Preview

Chapter 2 considers the case of the zero-sum Nash game. A composite control is found from two subsystems of the full system found by using singular perturbation techniques similar to those in [16] which is an $0(\mu)$ approximation of the optimal Nash control and leads to an $0(\mu^2)$ approximation of the optimal performance cost. The composite control formed satisfies the "asymptotic Nash" strategy concept. These results can also be found in [21].

Chapter 3 considers the nonzero-sum Nash game. It is shown that the nonzero-sum Nash game is ill-posed with respect to the usual reduction procedure of singular perturbation. If the perturbation is due to inadequately modeled fast dynamics, a reformulated performance index is

proposed which leads to a well-posed problem with respect to the usual singular perturbation reduction technique. A control is formed which is $O(\mu)$ to the optimal Nash control based on the reduced system involving only slow modes. If the perturbation is due to accurately modeled fast dynamics a modified reduction procedure is presented which transfers fast information to the slow game. A composite control is formed from modified slow game and fast game calculations which approximates the optimal Nash control to $O(\mu)$. This composite also leads to an $O(\mu)$ approximation of the optimal Nash cost. It is shown that the controls designed in this chapter, whether the perturbation is due to accurately or inaccurately modeled fast dynamics, satisfy the "asymptotic Nash" criteria.

Chapter 4 considers a load frequency example consisting of a two area power system. The two areas are identical and consist of a simple non-reheat steam turbine generator. There is also a tie-line between the two areas. Controls are designed and state trajectories shown, to illustrate the results of Chapter 3.

2. SINGULARLY PERTURBED ZERO-SUM NASH GAME

2.1. Introduction

For control problems of high dimension, a major difficulty in computing the optimal control is that the dimension of the Riccati equation is large. In game problems this "curse of dimensionality" is even more pronounced. In this chapter we deal with a zero-sum Nash game on a singularly perturbed system. The zero-sum game was chosen for study first because it has the property that due to the fact that the performance indices add to zero, the order of the Riccati equations is reduced and this allows us to extend the singular perturbation theory for control problems more readily than to a nonzero-sum game.

This chapter presents a design method to generate a "composite" control based on subsystem calculations which is an $O(\mu)$ approximation of the Nash optimal control and which yields an $O(\mu^2)$ approximation of the optimal cost. The advantage of this composite control is that it is computed from two reduced order systems. Also the value of the perturbation parameter is not needed in the design of the controls. Moreover, since the design is based on separate fast and slow problems the numerical stiffness of the full order problem is relieved. The design in this chapter is closely based on the work of Chow and Kokotovic [16].

2.2. Problem Statement

Consider a singularly perturbed time invariant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u_1 + B_{12}u_2, \qquad x_1(0) = x_{10} \tag{2.1a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2, \qquad x_2(0) = x_{20} \tag{2.1b}$$

where x_1, x_2, u_1, and u_2 are n_1, n_2, m_1, and m_2 vectors respectively and μ is a small positive scalar. Players one and two control u_1 and u_2 respectively which they choose based on a zero-sum linear quadratic Nash criterion given by

$$J_1 = -J_2 = \tfrac{1}{2}\int_0^{\infty} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_1'R_1u_1 + u_2'R_2u_2 \right) dt. \qquad (2.2)$$

The usual definiteness assumptions are made on R_i and on

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2' & Q_3 \end{bmatrix}.$$

In addition, we require that

$$\begin{bmatrix} \hat{R}_1 & \hat{Q}_3 \\ \hat{Q}_3' & \hat{R}_2 \end{bmatrix} \qquad (2.3)$$

where

$$\hat{R}_i = R_i + B_{2i}'(A_{22}^{-1})'Q_3A_{22}^{-1}B_{2i}$$
$$\hat{Q}_3 = B_{21}'(A_{22}^{-1})'Q_3A_{22}^{-1}B_{22}$$

be nonsingular.

Player one desires to minimize J_1 while player two desires to minimize J_2. Since $J_1 + J_2 = 0$ players one and two are in direct conflict. Any improvement in the value of J_1 for player one constitutes a worsening of the value of J_2 for player two.

The Riccati equations for (2.2) subject to (2.1) are well known [18] and are given by

$$0 = -A'S - SA - Q + S[B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2']S, \qquad (2.4)$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ \frac{B_{22}}{\mu} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{12} \\ \frac{B_{22}}{\mu} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The Nash controls are given by

$$u_i = -R_i^{-1} B_i' S x \tag{2.5}$$

where S is a stabilizing solution of (2.4).

The system (2.1) is a singularly perturbed system and can be shown to have two sets of eigenvalues $o(1)$ and $0(\frac{1}{\mu})$. Because of the separation in the eigenvalues the Riccati equation (2.4) is "numerically stiff." That is, because μ appears in A and B_i the Riccati gain is dependent on μ. When μ is small it can take a very sophisticated computer routine to get a solution to converge. We propose to take advantage of the two-time scale property of (2.1) to get a slow and a fast subsystem to approximate the slow and fast modes of (2.1) and then design a control using techniques similar to those in [16].

2.3. Slow and Fast Problem Formulation

We get the slow subsystem of (2.1) by formally setting $\mu = 0$ in (2.1b) and solving for x_2 assuming that A_{22} is nonsingular. This gives

$$\bar{x}_2 = -A_{22}^{-1}[A_{21}\bar{x}_1 + B_{21}\bar{u}_1 + B_{22}\bar{u}_2] \tag{2.6}$$

where the bar indicates that $\mu = 0$. Substituting $\bar{x}_2$ for x_2 in (2.1a) we get the slow subsystem, letting $\bar{x}_1 = x_s$, $\bar{u}_i = u_{is}$,

$$\dot{x}_s = A_o x_s + B_{01} u_{1s} + B_{02} u_{2s}, \qquad x_s(0) = x_{10} \tag{2.7}$$

where

$$A_o = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$B_{0i} = B_{1i} - A_{12}A_{22}^{-1}B_{2i} \quad , \quad i=1,2.$$

To get the corresponding slow performance indices we substitute (2.6) for x_2 and let $\mu \to 0$ in u_i and x_1. That is, replace u_i with u_{is} and x_1 by x_s. This gives

$$J_{1s} = -J_{2s} = \tfrac{1}{2}\int_0^\infty (x_s'\hat{Q}_1x_s + 2x_s'\hat{Q}_2(B_{21}u_{1s} + B_{22}u_{2s}) + u_{1s}'\hat{R}_1u_{1s} + u_{2s}'\hat{R}_2u_{2s} + 2u_{1s}'\hat{Q}_3u_{2s})dt \qquad (2.8)$$

where

$$\hat{Q}_1 = Q_1 - Q_2A_{22}^{-1}A_{21} - (A_{22}^{-1}A_{21})'Q_2' + (A_{22}^{-1}A_{21})'Q_3A_{22}^{-1}A_{21}$$

$$\hat{Q}_2 = (A_{22}^{-1}A_{21})'Q_3A_{22}^{-1} - Q_2A_{22}^{-1}$$

and $\hat{R}_i$ and $\hat{Q}_3$ are as defined earlier in this chapter. For (2.8) subject to (2.7) the Nash controls are easily found to be

$$\begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix} = -\begin{bmatrix} \hat{R}_1 & \hat{Q}_3 \\ \hat{Q}_3' & \hat{R}_2 \end{bmatrix}^{-1} \begin{bmatrix} B_{21}'\hat{Q}_2' + B_{01}'K_s \\ B_{22}'\hat{Q}_2' + B_{02}'K_s \end{bmatrix} x_s \qquad (2.9)$$

where K_s is a stabilizing solution of

$$0 = -\hat{Q}_1 - K_sA_o - A_o'K_s + \begin{bmatrix} B_{21}'\hat{Q}_2' + B_{01}'K_s \\ B_{22}'\hat{Q}_2' + B_{02}'K_s \end{bmatrix}' \begin{bmatrix} \hat{R}_1 & \hat{Q}_3 \\ \hat{Q}_3' & \hat{R}_2 \end{bmatrix}^{-1} \begin{bmatrix} B_{21}'\hat{Q}_2' + B_{01}'K_s \\ B_{22}'\hat{Q}_2' + B_{02}'K_s \end{bmatrix}. \qquad (2.10)$$

To derive the "fast" part of (2.1) corresponding to the large eigenvalues, let the slow variables be constant during the fast transient. Defining $x_f = x_2 - \bar{x}_2$, $u_{1f} = u_1 - u_{1s}$ and $u_{2f} = u_2 - u_{2s}$ the fast part of (2.1) is

$$\mu\dot{x}_f = A_{22}x_f + B_{21}u_{1f} + B_{22}u_{2f} \qquad , \; x_f(0) = x_{20} - \bar{x}_2(0). \qquad (2.11)$$

The fast performance index is then

$$J_{1f} = -J_{2f} = \tfrac{1}{2}\int_0^\infty (x_f' Q_3 x_f + u_{1f}' R_1 u_{1f} + u_{2f}' R_2 u_{2f})dt. \qquad (2.12)$$

The control for (2.12) subject to (2.11) is easily found to be

$$u_{if} = -R_i^{-1} B_{2i}' K_f x_f \qquad , \; i = 1,2 \qquad (2.13)$$

where

$$0 = -Q_3 - K_f A_{22} - A_{22}' K_f + K_f (B_{21} R_1^{-1} B_{21}' + B_{22} R_2^{-1} B_{22}') K_f. \qquad (2.14)$$

2.4. Composite Control Formulation

We will now design a control which is close to the optimal control (2.5) utilizing the fast and slow controls found in the previous section. If we let

$$u_{if} = G_{2i} x_f \qquad (2.15)$$

$$u_{is} = G_{0i} x_s \qquad (2.16)$$

where G_{2i} and G_{0i} are designed based on some criteria then the control $\tilde{u}_i = u_{is} + u_{if}$ can be written as

$$\tilde{u}_i = ((I + G_{2i} A_{22}^{-1} B_{2i}) G_{0i} + G_{2i} A_{22}^{-1} A_{21} + G_{2i} A_{22}^{-1} B_{2j} G_{0j}) x_s$$

$$+ G_{2i}(-A_{22}^{-1}(A_{21} + B_{2i} G_{0i} + B_{2j} G_{0j}) x_s + x_f), \qquad i,j = 1,2, \; j \neq i. \qquad (2.17)$$

Approximating x_s by x_1 and the last term in (2.17) by x_2 we have the following lemma.

Lemma 2.1: If the controls

$$u_{is} = G_{0i}x_s \qquad , \; i,j=1,2, \quad i \neq j \tag{2.18}$$

$$u_{if} = G_{2i}x_f \qquad , \; i,j=1,2, \quad i \neq j \tag{2.19}$$

$$\begin{aligned} u_{ic} = ((I + G_{2i}A_{22}^{-1}B_{2i})G_{0i} + G_{2i}A_{22}^{-1}A_{21} + G_{2i}A_{22}^{-1}B_{2j}G_{0j})x_1 \\ + G_{2i}x_2 \qquad , \; i,j=1,2, \quad i \neq j \end{aligned} \tag{2.20}$$

are applied to systems (2.7), (2.11), and (2.1) respectively and if $(A_{22} + B_{21}G_{21} + B_{22}G_{22})$ is stable then for all finite $t > 0$, then

$$x_1(t) = x_s(t) + 0(\mu) \tag{2.21}$$

$$x_2(t) = -A_{22}^{-1}(A_{21} + B_{21}G_{01} + B_{22}G_{02})x_s(t) + x_f(t) + 0(\mu) \tag{2.22}$$

$$u_{ic}(t) = u_{is}(t) + u_{if}(t) + 0(\mu) \qquad , \; i=1,2. \tag{2.23}$$

If in addition $[A_0 + B_{01}G_{01} + B_{02}G_{02}]$ is stable, (2.21)-(2.23) hold for all $t \in [0,\infty)$.

Proof: See Appendix A.

As in the control case, Lemma 1 suggests that G_{0i} and G_{2i}, $i=1,2$, can be designed separately and then used in a composite control defined by (2.20). We will consider the composite control to be given by (2.20) with G_{01} from (2.9) and G_{2i} from (2.13). The composite control can then be written as

$$\begin{aligned} \begin{bmatrix} u_{1c} \\ u_{2c} \end{bmatrix} = -\begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} R_1 - B_{21}'K_fA_{22}^{-1}B_{21} & -B_{21}'K_fA_{22}^{-1}B_{22} \\ -B_{22}'K_fA_{22}^{-1}B_{21} & R_2 - B_{22}'K_fA_{22}^{-1}B_{22} \end{bmatrix} \begin{bmatrix} \hat{R}_1 & \hat{Q}_3 \\ \hat{Q}_3' & \hat{R}_2 \end{bmatrix}^{-1} \\ \cdot \begin{bmatrix} B_{21}'\hat{Q}_2' + B_{01}'K_s \\ B_{22}'\hat{Q}_2' + B_{02}'K_s \end{bmatrix} x_1 - \begin{bmatrix} R_1^{-1}B_{21}'K_fA_{22}^{-1}A_{21} \\ R_2^{-1}B_{22}'K_fA_{22}^{-1}A_{21} \end{bmatrix} x_1 - \begin{bmatrix} R_1^{-1}B_{21}'K_f \\ R_2^{-1}B_{22}'K_f \end{bmatrix} x_2 . \end{aligned} \tag{2.24}$$

We would like to compare the composite control (2.24) and the Nash optimal control (2.5). To do this we need some relationships between K_s, K_f, and S.

Theorem 2.1: If S possesses a power series expansion at $\mu = 0$, i.e.

$$S = \begin{bmatrix} S_1 & \mu S_2 \\ \mu S_2' & \mu S_3 \end{bmatrix} + \sum_{j=1}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} S_1^{(j)} & \mu S_2^{(j)} \\ \mu S_2'^{(j)} & \mu S_3^{(j)} \end{bmatrix} \tag{2.25}$$

which implies that the slow game (2.7), (2.8) and the fast game (2.11), (2.12) possess unique stabilizing solutions, then

$$S_1 = K_s \tag{2.26}$$

$$S_2 = [-Q_2 - A_{21}'K_f - K_sA_{12} + K_s(B_{11}R_1^{-1}B_{21}' + B_{12}R_2^{-1}B_{22}')K_f] \cdot \\ \cdot [A_{22} - B_{21}R_1^{-1}B_{21}'K_f - B_{22}R_2^{-1}B_{22}'K_f]^{-1} \tag{2.27}$$

$$S_3 = K_f. \tag{2.28}$$

Proof: See Appendix B.

Theorem 2.1 demonstrates that the zero-sum Nash game is well-posed with respect to the usual singular perturbation techniques of optimal control theory. That is, (2.26) indicates that the performance cost of the full problem tends to the performance cost of the reduced or slow subsystem as $\mu \to 0^+$.

We would now like to compare the composite control (2.24) with the full order Nash control (2.5).

Theorem 2.2: If S possesses a power series expansion at $\mu = 0$ then

$$u_{ic} = u_i + 0(\mu). \tag{2.29}$$

Proof: See Appendix C.

Since we have the composite optimal Nash controls equal at $\mu = 0$ we may write the composite control as

$$u_{ic} = -R_i^{-1}B_i' \begin{bmatrix} K_s & 0 \\ \mu K_m' & \mu K_f \end{bmatrix} x \qquad (2.30a)$$

$$= -R_i^{-1}B_i'M_c x \qquad (2.30b)$$

where

$$K_m = S_2 = [-Q_2 - A_{21}'K_f - K_sA_{12} + K_s(B_{11}R_1^{-1}B_{21}' + B_{12}R_2^{-1}B_{22}')K_f] \cdot$$
$$\cdot [A_{22} - B_{21}R_1^{-1}B_{21}'K_f - B_{22}R_2^{-1}B_{22}'K_f]^{-1}. \qquad (2.31)$$

If the composite control (2.30) is applied to the full order system (2.1) for performance indices (2.2) a suboptimal cost results which we write as

$$J_{1c} = -J_{2c} = \tfrac{1}{2}x_o'P_cx_o \qquad (2.32)$$

where P_c is a stabilizing solution of

$$0 = P_c[A-(B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2')M_c] + [A-(B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2')M_c]'P_c$$
$$+ Q + M_c'(B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2')M_c. \qquad (2.33)$$

We would now like to compare the costs resulting from the composite control application (2.32) and the Nash optimal control application

$$J_1 = -J_2 = \tfrac{1}{2}x_o'Sx_o. \qquad (2.34)$$

<u>Theorem 2.3</u>: If P_c and S have power series expansions about $\mu = 0$, i.e.

$$P_c = \sum_{i=0}^{\infty} \frac{\mu^i}{i!} \begin{bmatrix} P_{c1}^{(i)} & \mu P_{c2}^{(i)} \\ \mu P_{c2}'^{(i)} & \mu P_{c3}^{(i)} \end{bmatrix}$$

$$S = \begin{bmatrix} S_1 & \mu S_2 \\ \mu S_2' & \mu S_3 \end{bmatrix} + \sum_{i=1}^{\infty} \frac{\mu^i}{i!} \begin{bmatrix} S_1^{(i)} & \mu S_2^{(i)} \\ \mu S_2'^{(i)} & \mu S_3^{(i)} \end{bmatrix}$$

then when u_{1c} and u_{2c} are applied to (2.1), (2.2)

$$J_{ic} = J_{i_{opt}} + 0(\mu^2). \tag{2.35}$$

<u>Proof</u>: See Appendix D.

Theorem 2.3 indicates that the composite controls lead to a close approximation of the optimal cost. However, we do not know in what sense the composite controls are related to the Nash equilibrium concept. To define the relationship of the composite controls to the Nash strategy concept we have the following theorem.

<u>Theorem 2.4</u>: The control pair (u_{1c},u_{2c}) satisfies the asymptotic saddle point condition

$$J_1(u_{1c},u_2)+0(\mu) \leq J_1(u_{1c},u_{2c}) \leq J_1({}_{1c},u_{2c})+0(\mu) \tag{2.36}$$

where u_i are in some allowed strategy set. (By allowed we mean that if controls from these sets are applied to (2.1), (2.2) the resulting cost matrix possesses a power series expansion at $\mu = 0$.)

<u>Proof</u>: See Appendix E.

(2.36) says that the set (u_{1c},u_{2c}) satisfies an asymptotic saddle-point condition. In this sense, the set (u_{1c},u_{2c}) is asymptotic to the Nash strategy set and in fact (2.36) puts limits on the maximum improvement possible if one player deviates unilaterally from the composite control strategy.

3. SINGULARLY PERTURBED NONZERO-SUM NASH GAME

3.1. Introduction

In this chapter we consider a singularly perturbed nonzero-sum Nash game. In general the nonzero-sum game is more difficult to solve than the zero-sum game because the order of the Riccati equations is higher and we have coupling between two sets of equations. The singular perturbation aspect of the problem complicates matters even more by introducing numerical "stiffness" which is even more troublesome in the higher order problems of the nonzero-sum Nash game than in the zero-sum Nash game.

The objective of this chapter is to examine the nonzero-sum Nash problem from both a formulational and a computational standpoint to derive near-optimum controls based on *lower*-order computations which yield near-optimal performance.

We give an example of a nonzero-sum Nash game whereby the natural singular perturbation leads to a strategy which results in limiting values of performance indices different from the limiting values of those corresponding to the full order Nash strategy. In contrast we have shown in Chapter 2 that the corresponding performance indices have the same limiting values when the game is zero-sum.

We then show that a physically justified modification of the performance indices consistent with inadequate modeling of fast dynamics results in a well-posed singularly perturbed nonzero-sum Nash game problem when the natural perturbation method is applied. With this modification, computational savings can be gained and a close approximation to the optimal performance indices obtained by order reduction of the Riccati equations.

Finally, we present a hierarchical reduction procedure which leads to a well-posed singularly perturbed modified slow game. This reduced order slow game differs from the natural one in that it contains information about the low order fast game. The problem is well-posed with respect to the original performance indices for the full order game. Computational savings and a close approximation of the performance indices are achieved.

3.2. Ill-Posedness of Nonzero-Sum Nash Games with Respect to Singular Perturbation

Consider a singularly perturbed time-invariant system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u_1 + B_{12}u_2 \quad ; \; x_1(t_0) = x_{10} \tag{3.1a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2 \quad ; \; x_2(t_0) = x_{20} \tag{3.1b}$$

and performance criteria

$$J_i = \frac{1}{2}\int_{t_0}^{\infty}\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'\begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_i'R_{ii}u_i + u_j'R_{ij}u_j\right\}dt \; ; \; i,j=1,2, \; i\neq j \tag{3.2}$$

where μ is a small positive scalar, x_1 and x_2 are n_1- and n_2-dimensional components of the state vector, u_1 and u_2 are m_1- and m_2-dimensional control vectors to be chosen by Players 1 and 2 respectively in accordance with the Nash solution concept, and the control strategies are restricted to be linear feedback functions of the state. Denote

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}/\mu & A_{22}/\mu \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1i} \\ B_{2i}/\mu \end{bmatrix}, \text{ and } Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}.$$

The usual definiteness assumptions are made on Q_i and R_{ij}, $i,j = 1,2$. Also R_{ii} and Q_{i3} are chosen so that

$$\begin{bmatrix} \hat{R}_{11} & \hat{Q}_{13} \\ \hat{Q}_{23} & \hat{R}_{22} \end{bmatrix}$$

is nonsingular where

$$\hat{R}_{ii} = R_{ii} + B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2i}$$
$$\hat{Q}_{i3} = B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j}.$$

The nonsingularity of this matrix is necessary for existence of the Nash controls both in the full game we define in the next section and for the slow game we will define in this section.

The optimal closed-loop Nash strategy for Player i for (3.2) subject to (3.1) is well known [18] and given by

$$u_i = -R_{ii}^{-1}B_i'K_ix \tag{3.3}$$

where K_i is a stabilizing solution of the coupled Riccati equations given by

$$0 = -(Q_i + K_iA + A'K_i) + K_iB_iR_{ii}^{-1}B_i'K_i + K_iB_jR_{jj}^{-1}B_j'K_j + K_jB_jR_{jj}^{-1}B_j'K_i$$
$$-K_jB_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'K_j, \qquad \text{for } i,j = 1,2;\ \ i \neq j. \tag{3.4}$$

Notice that since A and B_i are functions of the small parameter μ, K_i is also a function of μ. In general even for low order problems the presence of μ causes numerical "stiffness" in (3.4). For this reason and for computational reduction the problem (3.1),(3.2) in the one player, i.e. control, case is generally approximated by a lower order problem by formally setting $\mu = 0$. This produces a control which when applied to the full order plant gives a close approximation to the optimal cost. In this section we examine the standard reduction method.

The standard approach to obtaining a reduced order model is to formally set $\mu = 0$ in (3.1b), solve for x_2, assuming A_{22} is nonsingular, and substitute in (3.1a) to obtain

$$\dot{\bar{x}}_1 = (A_{11}-A_{12}A_{22}^{-1}A_{21})\bar{x}_1 + (B_{11}-A_{12}A_{22}^{-1}B_{21})\bar{u}_1 + (B_{12}-A_{12}A_{22}^{-1}B_{22})\bar{u}_2, \quad \bar{x}_1(t_0) = x_{10} \tag{3.5}$$

where the bar indicates that $\mu = 0$. Rewriting (3.5) we get the "slow" (since setting $\mu = 0$ is equivalent to saying that the fast states are infinitely fast) subsystem

$$\dot{x}_s = A_0 x_s + B_{01}u_{1s} + B_{02}u_s \qquad , \; x_s(t_0) = x_{10} \tag{3.6}$$

where

$$A_0 = A_{11}-A_{12}A_{22}^{-1}A_{21},$$

$$B_{0i} = B_{1i}-A_{12}A_{22}^{-1}B_{2i} \qquad , \; i = 1,2.$$

The corresponding "slow" performance criteria found by substituting x_2 when $\mu = 0$ into (3.2) is

$$J_{is} = \frac{1}{2}\int_{t_0}^{\infty} \{x_s'\hat{Q}_{i1}x_s + 2x_s'\hat{Q}_{i2}(B_{2i}u_{is}+B_{2j}u_{js}) + u_{is}'\hat{R}_{ii}u_{is} + u_{js}'\hat{R}_{ij}u_{js} + 2u_{is}'\hat{Q}_{i3}u_{js}\} \qquad i,j = 1,2, \quad i \neq j \tag{3.7}$$

where

$$\hat{Q}_{i1} = Q_{i1}-Q_{i2}A_{22}^{-1}A_{21}-(A_{22}^{-1}A_{21})'Q_{i2}' + (A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}A_{21},$$

$$\hat{Q}_{i2} = (A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}-Q_{i2}A_{22}^{-1},$$

$$\hat{R}_{ii} = R_{ii} + B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2i},$$

$$\hat{R}_{ij} = R_{ij} + B_{2j}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j},$$

$$\hat{Q}_{i3} = B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j}.$$

Solving for the reduced order closed-loop Nash strategies, we have

$$u_{is} = -\hat{R}_{ii}^{-1}[B_{0i}'K_{is}x_s + B_{2i}'\hat{Q}_{i2}'x_s + \hat{Q}_{i3}u_{js}] \tag{3.8a}$$

$$= -M_{is}x_s \tag{3.8b}$$

where K_{is} is a stabilizing solution of

$$0 = -(\hat{Q}_{i1} + A_0'K_{is} + K_{is}A_0) + M_{is}'\hat{R}_{ii}M_{is} - M_{js}'\hat{R}_{ij}M_{js} + [K_{is}B_{0j} + \hat{Q}_{i2}B_{2j}]M_{js}$$

$$+ M_{js}'[B_{0j}'K_{is} + B_{2j}'\hat{Q}_{i2}'], \quad \text{for } i,j = 1,2, \ i \neq j. \tag{3.9}$$

Using the gain matrix M_{is} from (3.8b), we implement the control

$$u_i = -M_{is}x_1 \tag{3.10}$$

and apply it to the system in (3.1). The resulting value of the suboptimal performance criteria in (3.2) can be expressed as

$$J_{i_{sub}} = \frac{1}{2}x'(t_0)V_{i_{sub}}x(t_0) \tag{3.11}$$

where $V_{i_{sub}}$ satisfies the Lyapunov equation

$$V_{i_{sub}}\{A - B_i[M_{is} \vdots 0] - B_j[M_{js} \vdots 0]\} + \{A - B_i[M_{is} \vdots 0] - B_j[M_{js} \vdots 0]\}'V_{i_{sub}}$$

$$\left[\begin{array}{c|c} Q_{i1} + M_{is}'R_{ii}M_{is} + M_{js}'R_{ij}M_{js} & Q_{i2} \\ \hline Q_{i2}' & Q_{i3} \end{array}\right] = 0. \tag{3.12}$$

The matrix $V_{i_{sub}}$ depends on μ since A and B_i contain μ. Hence the reduced cost is dependent on μ.

If the optimal Nash controls given by (3.3) are applied to (3.1), the values of the optimal performance criteria are given by

$$J_i = \frac{1}{2}x'(t_0)K_ix(t_0) \tag{3.13}$$

where K_i satisfies (3.4). We wish to examine the nature of the optimal

criteria J_i as $\mu \to 0$. In particular we wish to verify if J_i approaches $J_{i_{sub}}$ as μ approaches zero. We will say that the reduced order game is well-posed if J_i approaches $J_{i_{sub}}$ as $\mu \to 0$. Otherwise, we say that it is ill-posed. We perform this comparison on a specific numerical example.

Consider the second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1/\mu & -2/\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2/\mu \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 2/\mu \end{bmatrix} u_2 \tag{3.14}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

with performance criteria

$$J_1 = \frac{1}{2}\int_0^\infty \{x'\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}x + u_1^2 + 2u_2^2\}dt \tag{3.15a}$$

$$J_2 = \frac{1}{2}\int_0^\infty \{x'\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}x + 2u_1^2 + u_2^2\}dt. \tag{3.15b}$$

For this example, the resulting K_{is} and M_{is} from (3.9) and (3.8) are

$$K_{1s} = K_{2s} = \sqrt{\frac{25}{54}} = .6804 \tag{3.16}$$

and

$$M_{1s} = M_{2s} = .4082. \tag{3.17}$$

Calculation of the resulting values of J_1 for several values of μ are given in Table 3.1. Because of symmetry, $J_1 = J_2 = J$. It is seen that the limit of J_i as $\mu \to 0$ is different from the corresponding limit of $J_{i_{sub}}$. This discrepancy between the J's in the neighborhood of $\mu = 0$ indicates that the reduced order Nash strategy obtained by the standard method is ill-posed.

Table 3.1. Comparison of optimal Nash and suboptimal costs for several values of μ

μ	.5	.2	.1	.01	.005	.001	0
J	1.3012	.73245	.5425	.3724	.3630	.3630	.3536
J_{sub}	1.84127	.86558	.59083	.36420	.35217	.34259	.3402

3.3. Regularization of the Cost Functional Consistent with Inadequate Modeling of Fast Dynamics

The manner in which the singular perturbation approach could be modified so that we have a well-posed problem depends on the reason for the appearance of the singular perturbation parameter in the system model. In this section we discuss the first of two reasons considered in this chapter. Let us suppose that we have a Nash strategy using a model represented by (3.1a) and

$$0 = A_{21}x_1 + A_{22}x_2 + B_{21}u_1 + B_{22}u_2. \tag{3.18}$$

We wish to examine the robustness of the Nash strategies when the actual system is represented by (3.1b) instead of (3.18). The performance index in (3.2) leads to an ill-posed problem as we demonstrated.

If indeed the original model used for design is based on (3.1a) and (3.18), then for consistency it is appropriate to assume that the vector x_2 that appears in (3.2) is constrained by (3.18). That is, from (3.18) we have

$$x_2 = -A_{22}^{-1}[A_{21}x_1 + B_{21}u_1 + B_{22}u_2]. \tag{3.19}$$

Substituting (3.19) into (3.1a) we obtain (3.5) and substituting (3.19) into (3.2) we obtain

$$J_i = \frac{1}{2}\int_{t_0}^{\infty} \{x_1'\hat{Q}_{i1}x_1 + 2x_1'\hat{Q}_{i2}(B_{2i}u_i + B_{2j}u_j) + u_i'\hat{R}_{ii}u_i + u_j'\hat{R}_{ij}u_j + 2u_i'\hat{Q}_{i3}u_j\}dt. \quad (3.20)$$

The modified performance index in (3.20) for $i,j = 1,2$, $i \neq j$, reflects the model constraint of (3.18). In this case, the variable x_2 in (3.2) is not a component vector of the state x, but it is simply a function of x_1, u_1, and u_2 as given in (3.19). For example, in a dc motor model, we may be interested in penalizing the armature current. However if our model neglects armature inductance then the armature current is expressed as a function of the speed and the voltage. On the other hand, in our earlier ill-posed example, x_2 in (3.2) is not constrained to satisfy (3.19) but instead, it is part of the state as given in (3.1b). Thus in this reformulated problem, we are interested in comparing the Nash strategy that is obtained from (3.1a), (3.18) and (3.20), which is the same as (3.6) and (3.7), with the Nash strategy that is obtained from (3.1a), (3.1b), and (3.20) as $\mu \to 0$. We show that this is a well-posed problem with respect to singular perturbation so that the Nash strategy is robust against inaccuracies caused by neglecting fast dynamics, provided that these are stable (i.e. A_{22} is stable).

For the full order problem (3.1) and (3.20), the optimal closed-loop Nash equilibrium solution is given by

$$u_i = -\hat{R}_{ii}^{-1}\{B_{2i}'[\hat{Q}_{i2}' \vdots 0]x + [B_{1i}' \vdots B_{2i}'/\mu]\bar{K}_i x + \hat{Q}_{i3}u_j\} \quad (3.21a)$$

$$= -\bar{M}_i x \quad (3.21b)$$

where $\overline{K}_i$ is a stabilizing solution of

$$0 = \begin{bmatrix} \hat{Q}_{i1} & 0 \\ 0 & 0 \end{bmatrix} + \overline{K}_i A + A'\overline{K}_i - \overline{K}_i B_j \overline{M}_j - \overline{M}_j' B_j' \overline{K}_i$$

$$- \begin{bmatrix} \hat{Q}_{i2} \\ 0 \end{bmatrix} B_{2j}\overline{M}_j - \overline{M}_j' B_{2j}' [\hat{Q}_{i2}' \vdots 0] + \overline{M}'_j \hat{R}_{ij} \overline{M}_j - \overline{M}_i' \hat{R}_{ii} \overline{M}_i . \tag{3.22}$$

The optimal cost for (3.20) subject to (3.1) is given by

$$J_{i_{opt}} = \frac{1}{2} x'(t_0)\overline{K}_i x(t_0). \tag{3.23}$$

If the reduced control

$$u_{ir} = -M_{is} x_1 \tag{3.24}$$

where M_{is} is from (3.8) is applied to (3.1) for performance indices (3.20) a suboptimal performance cost results which can be written as

$$V_{ir} = \frac{1}{2} x'(t_0) P_{ir} x(t_0) \tag{3.25}$$

where P_{ir} is the positive semidefinite solution of the Lyapunov equation

$$P_{ir}\{A - B_i[M_{is} \vdots 0] - B_j[M_{js} \vdots 0]\} + \{A - B_i[M_{is} \vdots 0] - B_j[M_{js} \vdots 0]\}' P_{ir} + \begin{bmatrix} \xi_i & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{3.26}$$

and

$$\xi_i = \hat{Q}_{i1} - \hat{Q}_{i2}[B_{2i}M_{is} + B_{2j}M_{js}] - [B_{2i}M_{is} + B_{2j}M_{js}]'\hat{Q}_{i2}' + M_{is}'\hat{R}_{ii}M_{is} + M_{js}'\hat{R}_{ij}M_{js}$$
$$+ M_{is}'\hat{Q}_{i3}M_{js} + M_{js}'\hat{Q}_{i3}'M_{is}.$$

We shall compare the optimal Nash performance cost (3.23) to the suboptimal performance cost (3.25). In order to perform this comparison we need some relationship between the optimal Riccati gain $\overline{K}_i$ and P_{ir}. A relationship is found by first giving conditions under which $\overline{K}_i$ possesses

a power series expansion $\mu = 0$ and then giving conditions under which P_{ir} possesses a power series expansion. Finally, we form a new Lyapunov equation by subtracting (3.26) from (3.22) and show that there is in fact a relationship between the optimal performance cost and the suboptimal performance cost.

Represent $\bar{K}_i$, the solution of (3.22) as

$$\bar{K}_i(\mu) = \begin{bmatrix} \bar{K}_{i1}(\mu) & \mu\bar{K}_{i2}(\mu) \\ \mu\bar{K}'_{i2}(\mu) & \mu\bar{K}_{i3}(\mu) \end{bmatrix}, \quad i = 1,2. \tag{3.27}$$

Substitution of (3.27) into (3.22) at $\mu = 0$ and partitioning gives the following algebraic equations.

$$\begin{aligned} 0 = {} & \hat{Q}_{i1} + \bar{K}_{i1}^{(0)}A_{11} + A'_{11}\bar{K}_{i1}^{(0)} + \bar{K}_{i2}^{(0)}A_{21} + A'_{21}\bar{K}'^{(0)}_{i2} \\ & - [\bar{K}_{i1}^{(0)}B_{1j} + \bar{K}_{i2}^{(0)}B_{2j} + \hat{Q}_{i2}B_{2j}][\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}S_{j1} \\ & - S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}[B'_{1j}\bar{K}_{i1}^{(0)} + B'_{2j}\bar{K}'^{(0)}_{i2} + B'_{2j}\hat{Q}'_{i2}] \\ & + S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}\hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}S_{j1} \\ & - S'_{i1}[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}\hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}S_{i1} \end{aligned} \tag{3.28}$$

$$\begin{aligned} 0 = {} & \bar{K}_{i1}^{(0)}A_{12} + \bar{K}_{i2}^{(0)}A_{22} + A'_{21}\bar{K}_{i3}^{(0)} \\ & - [\bar{K}_{i1}^{(0)}B_{1j} + \bar{K}_{i2}^{(0)}B_{2j} + \hat{Q}_{i2}B_{2j}][\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}[B'_{2j}\bar{K}_{j3}^{(0)} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}B'_{2i}\bar{K}_{i3}^{(0)}] \\ & - S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}B'_{2j}\bar{K}_{i3}^{(0)} \\ & + S'_{j1}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}\hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}[B'_{2j}\bar{K}_{j3}^{(0)} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}B'_{2i}\bar{K}_{i3}^{(0)}] \\ & - S'_{i1}[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}\hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}[B'_{2i}\bar{K}_{i3}^{(0)} - \hat{Q}_{j3}\hat{R}_{jj}^{-1}B'_{2j}\bar{K}_{j3}^{(0)}] \end{aligned} \tag{3.29}$$

$$0 = \bar{K}_{i3}^{(0)}A_{22} + A_{22}'\bar{K}_{i3}^{(0)} - \bar{K}_{i3}^{(0)}B_{2j}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}(B_{2j}'\bar{K}_{j3}^{(0)} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}B_{2i}'\bar{K}_{i3}^{(0)})$$

$$- (\bar{K}_{j3}^{(0)}B_{2j} - \bar{K}_{i3}^{(0)}B_{2i}\hat{R}_{ii}^{-1}\hat{Q}_{j3}')[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}B_{2j}'\bar{K}_{i3}^{(0)}$$

$$+ (\bar{K}_{j3}^{(0)}B_{2j} - \bar{K}_{i3}^{(0)}B_{2i}\hat{R}_{ii}^{-1}\hat{Q}_{j3}')[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}\hat{R}_{ij}[\hat{R}_{jj} - \hat{Q}_{j3}\hat{R}_{ii}^{-1}\hat{Q}_{i3}]^{-1}(B_{2j}'\bar{K}_{j3} -$$

$$- \hat{Q}_{j3}\hat{R}_{ii}^{-1}B_{2i}'\bar{K}_{i3}^{(0)})$$

$$- (\bar{K}_{i3}^{(0)}B_{2i} - \bar{K}_{j3}^{(0)}B_{2j}\hat{R}_{jj}^{-1}\hat{Q}_{i3}')[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}\hat{R}_{ii}[\hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}]^{-1}\cdot$$

$$\cdot(B_{2i}'\bar{K}_{i3}^{(0)} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}B_{2j}'\bar{K}_{j3}^{(0)}) \tag{3.30}$$

where

$$S_{i1} = B_{2i}'\hat{Q}_{i2}' + B_{1i}'\bar{K}_{i1}^{(0)} + B_{2i}'\bar{K}_{i2}'^{(0)} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}[B_{2j}'\hat{Q}_{j2}' + B_{1j}'\bar{K}_{j1}^{(0)} + B_{2j}'\bar{K}_{j2}'^{(0)}] \tag{3.31}$$

and

$$\bar{K}_{ik}^{(0)} = \bar{K}_{ik}(\mu)\big|_{\mu=0} \quad , \ i=1,2, \ \ k=1,2,3, \ \ j=1,2, \ \ i \neq j. \tag{3.32}$$

In the comparison of the optimal performance and the suboptimal performance costs we need the following conditions:

Condition a:

$$\begin{bmatrix} \mathcal{a}_1 & \mathcal{B}_1 \\ \mathcal{B}_2 & \mathcal{a}_2 \end{bmatrix} \tag{3.33}$$

is nonsingular where

$$\mathcal{a}_i = I \otimes \hat{A}_i' + \hat{A}_i' \otimes I$$

$$\mathcal{B}_i = I \otimes \hat{B}_i' + \hat{B}_i' \otimes I$$

and

$$\hat{A}_i = A_0 - B_{0j}M_{js} + B_{0i}\hat{R}_{ii}^{-1}\hat{Q}_{i3}'\tilde{R}_{jj}^{-1}[B_{2j}'\hat{Q}_{i2}' + B_{0j}'K_{is} - \hat{R}_{ij}M_{js}] - B_{0i}\tilde{R}_{ii}^{-1}\hat{R}_{ii}M_{is}$$

$$\hat{B}_i = B_{0j}\tilde{R}_{jj}^{-1}[B_{2j}'\hat{Q}_{i2}' + B_{0j}'K_{is} - \hat{R}_{ij}M_{js}] + B_{0j}\hat{R}_{jj}^{-1}\hat{Q}_{i3}'\tilde{R}_{ii}^{-1}\hat{R}_{ii}M_{is}$$

where

$$\tilde{R}_{ii} = \hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3} \qquad , \ i,j=1,2, \ \ i \neq j.$$

$\otimes$ is the Kronecker product operator. This condition is to guarantee existence and uniqueness of the solution of a set of coupled Lyapunov equations and is found in the proof of Theorem 3.1.

<u>Theorem 3.1</u>: If

1) A_{22} from (3.1) is stable,
2) the slow game (3.6), (3.7) has a unique stabilizing closed-loop Nash strategy pair,
3) $\bar{K}_{i3}^{(0)} = 0$ is the unique positive semidefinite solution of (3.30),

and 4) Condition a is satisfied,

then the solution $\bar{K}_i = \bar{K}_i(\mu)$ of (3.22) possesses a power series expansion at $\mu = 0$, that is,

$$\bar{K}_i(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} \bar{K}_{i1}^{(j)} & \mu\bar{K}_{i2}^{(j)} \\ \mu\bar{K}_{i2}^{'(j)} & \mu\bar{K}_{i3}^{(j)} \end{bmatrix} \tag{3.34}$$

where

$$\bar{K}_{ik}^{(j)} = \left.\frac{\partial^j \bar{K}_{ik}(\mu)}{\partial \mu^j}\right|_{\mu=0}, \qquad i = 1,2; \quad k = 1,2,3. \tag{3.35}$$

Furthermore, the matrices $\bar{K}_{i1}^{(0)}$, $\bar{K}_{i2}^{(0)}$, and $\bar{K}_{i3}^{(0)}$ satisfy the identities

$$\bar{K}_{i1}^{(0)} = K_{is} \tag{3.36a}$$

$$\bar{K}_{i2}^{(0)} = -K_{is}A_{12}A_{22}^{-1} \tag{3.36b}$$

$$\bar{K}_{i3}^{(0)} = 0. \tag{3.36c}$$

<u>Proof</u>: The proof is given in Appendix F.

A relationship between the suboptimal control (3.24) and the optimal control (3.21) is found by substituting (3.34) into (3.21) and

letting $\mu = 0$. Comparison of the resulting equation and (3.24) using the identities (3.36) yields

$$u_{ir} = u_i + 0(\mu) \qquad , \; i = 1,2. \tag{3.37}$$

The result in (3.37) is analogous to the "composite" control formulation in [16]. Even though there is no x_2 present in (3.24), the result in (3.37) is not unexpected since the fast part of $\overline{K}_i$ for $\mu = 0$ is zero. Thus we really have a "composite" control but the fast part of that composite control is zero.

Since the reduced control (3.24) is close to the optimal control (3.21) we expect that the reduced cost (3.25) is close to the optimal cost (3.23). We state the following results.

Theorem 3.2: If A_{22} is stable and if there exists a unique stabilizing closed-loop Nash solution to the slow game (3.6), (3.7), then P_{ir}, $i = 1,2$, in (3.25), (3.26) possesses a power series expansion at $\mu = 0$, that is,

$$P_{ir}(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} P_{i1}^{(j)} & \mu P_{i2}^{(j)} \\ \mu P_{i2}'^{(j)} & \mu P_{i3}^{(j)} \end{bmatrix} \tag{3.38}$$

where

$$P_{ir}^{(j)} = \left. \frac{\partial^j P_{ik}(\mu)}{\partial \mu^j} \right|_{\mu=0} \qquad , \; i = 1,2, \quad k = 1,2,3.$$

Proof: See Appendix G.

Applying the reduced control (3.24) to the system (3.1) and comparing the resulting cost to the optimal cost gives the following theorem.

<u>Theorem 3.3</u>: The first terms of the power series expansion at $\mu = 0$ of $J_{i_{opt}}$ in (3.23) and V_{ir} in (3.25) are the same, that is,

$$V_{ir} = J_{i_{opt}} + 0(\mu) \qquad , \; i = 1,2. \tag{3.39}$$

Thus the reduced order slow game in (3.6) and (3.7) is well-posed with respect to the normal singular perturbation reduction method for the full order game in (3.1) and (3.20).

<u>Proof</u>: See Appendix H.

It should be noted that in (3.20) x_2 does not appear explicitly and that there are cross terms of x_1 with u_1 and u_2 and also cross terms of u_1 and u_2. Using a linear transformation among x_1, u_1 and u_2, a performance criterion without cross terms could be obtained. However, in this case, the transformation would induce an additional structural constraint on the control, and the Nash solution might be different. Thus, no such transformation is used in this section. A second point to note is that although x_2 does not appear in (3.20), the "slow" part of x_2 as given by (3.19) does appear, since (3.19) was substituted into (3.2) to obtain (3.20).

Theorem 3.3 shows the closeness of the performance indices when a approximation of the Nash controls are used instead of the Nash controls themselves. However, we are concerned in what sense the composite controls are related to the Nash equilibrium concept. For instance, we know that it may be possible to find a control set which gives a good approximation of the Nash cost but not satisfy the Nash criteria. The following theorem comments on the Nash relationship of the composite controls.

<u>Theorem 3.4</u>: The reduced control pair (u_{1r}, u_{2r}) satisfies the asymptotic Nash relationships

$$J_1(u_{1r},u_{2r}) \le J_1(u_1,u_{2r}) + O(\mu) \tag{3.40a}$$

$$J_2(u_{1r},u_{2r}) \le J_2(u_{1r},u_2) + O(\mu) \tag{3.40b}$$

where u_i are in some allowed strategy set.

Proof: The proof is similar to the proof of Theorem 2.4 and is omitted for brevity.

Theorem 3.4 indicates that if one player deviates unilaterally from his composite control he can at most improve his performance cost by $O(\mu)$. Of course, there is no guarantee that one player might not deviate from the composite strategy since we no longer have the strictness inherent in the Nash equilibrium strategy. However, if μ is sufficiently small there would be little incentive to cheat.

3.4. Hierarchical Reduction Scheme Which Transfers Fast Game Information to a Modified Slow Game

In Section 3.3 we demonstrated that if the system model for control design contains only slow modes and we wish to determine the robustness of the nonzero-sum Nash strategy to the presence of fast modes in the actual system, then the performance indices should not include the fast modes of the system. That is, if we have a system with fast and slow modes, then in order to have a well-posed reduced problem under the usual singular perturbation reduction method, the fast modes of the system should not be penalized in the performance indices. On the other hand, if the system is assumed to be adequately modeled and the fast modes appear in both the state equations and performance indices, and it is desired to reduce the amount of computation and alleviate the numerical stiffness of the closed-loop Nash control

problem, we have seen via the example in Section 3.2 that the usual order reduction method of singularly perturbed optimal control problems does not lead to a well-posed Nash game.

In the method of Section 3.2 it is implicitly assumed that the fast modes and slow modes can be completely decoupled. However, we have shown that if we directly penalize the fast modes, the fast and slow modes cannot be completely decoupled. Taking this into account we propose to first solve a fast low order game and them implement the fast feedback control in the system and performance indices before obtaining a reduced order slow game. Thus we are proposing a block triangular or hierarchical rather than the usual block diagonal decomposition.

To derive the fast subsystem, we assume that the slow variables are constant during fast transients. Denoting the fast variables by the subscript f we have the fast subsystem and performance indices

$$\mu\dot{x}_f = A_{22}x_f + B_{21}u_{1f} + B_{22}u_{2f} \qquad ; \; x_f(t_0) = x_{20} - \bar{x}_2(t_0) \tag{3.41a}$$

$$J_{if} = \frac{1}{2}\int_{t_0}^{\infty} [x_f'Q_{i3}x_f + u_{if}'R_{ii}u_{if} + u_{jf}'R_{ij}u_{jf}]dt \qquad ; \; i,j=1,2; \; i\neq j \tag{3.41b}$$

where $x_f = x_2 - x_{2s}$ and $\bar{x}_2$ is found from (3.47). The closed-loop Nash controls for (3.41b) subject to (3.41a) are

$$u_{if} = -R_{ii}^{-1}B_{2i}'K_{if}x_f \qquad , \; i=1,2 \tag{3.42}$$

where K_{if} is a stabilizing solution of

$$\begin{aligned} 0 = &-Q_{i3} - K_{if}A_{22} - A_{22}'K_{if} + K_{if}B_{2i}R_{ii}^{-1}B_{2i}'K_{if} + K_{if}B_{2j}R_{jj}^{-1}B_{2j}'K_{jf} \\ &+ K_{jf}B_{2j}R_{jj}^{-1}B_{2j}'K_{if} - K_{jf}B_{2j}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf} \qquad , \; i,j=1,2, \; i\neq j. \end{aligned} \tag{3.43}$$

Next we make use of the fast control and substitute the following for u_i in our original system (3.1) and performance indices (3.2). Let

$$u_i = -R_{ii}^{-1}B_{2i}'K_{if}x_2 + \hat{u}_i \tag{3.44}$$

be our modified control. This gives a new system and performance indices given by

$$\dot{x}_1 = A_{11}x_1 + \hat{A}_{12}x_2 + B_{11}\hat{u}_1 + B_{12}\hat{u}_2 \qquad ; \ x_1(t_0) = x_{10} \tag{3.45a}$$

$$\mu\dot{x}_2 = A_{21}x_1 + \hat{A}_{22}x_2 + B_{21}\hat{u}_1 + B_{22}\hat{u}_2 \qquad ; \ x_2(t_0) = x_{20} \tag{3.45b}$$

and

$$J_i = \frac{1}{2}\int_{t_0}^{\infty}\{x'\begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & \tilde{Q}_{i3}\end{bmatrix}x - 2x_2'K_{if}B_{2i}\hat{u}_i - 2x_2'K_{jf}B_{2j}R_{jj}^{-1}R_{ij}\hat{u}_j$$

$$+ \hat{u}_i'R_{ii}\hat{u}_i + \hat{u}_j'R_{ij}\hat{u}_j\}dt \qquad i,j = 1,2, \quad i \neq j, \tag{3.46}$$

where

$$\hat{A}_{12} = A_{12} - B_{11}R_{11}^{-1}B_{21}'K_{1f} - B_{12}R_{22}^{-1}B_{22}'K_{2f}$$

$$\hat{A}_{22} = A_{22} - B_{21}R_{11}^{-1}B_{21}'K_{1f} - B_{22}R_{22}^{-1}B_{22}'K_{2f}$$

$$\tilde{Q}_{i3} = Q_{i3} + K_{if}B_{2i}R_{ii}^{-1}B_{2i}'K_{if} + K_{jf}B_{2j}R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}.$$

To get out "modified slow" subsystem we formally set $\mu = 0$ in (3.45b) and solve for x_2. This gives

$$\bar{x}_2 = -\hat{A}_{22}^{-1}[A_{21}\bar{x}_1 + B_{21}\bar{u}_1 + B_{22}\bar{u}_2]. \tag{3.47}$$

Substitution of (3.47) into (3.45a) and (3.46) gives us the "modified slow" subsystem and performance indices

$$\dot{x}_{sm} = \tilde{A}_0x_{sm} + \tilde{B}_{01}u_{1sm} + \tilde{B}_{02}u_{2sm} \qquad ; \ x_{sm}(t_0) = x_{10} \tag{3.48}$$

and

$$J_{ism} = \frac{1}{2}\int_{t_0}^{\infty}[x'_{sm}\tilde{Q}_{i1}x_{sm} + 2x'_{sm}\tilde{Q}_{i2}u_{ism} + 2x'_{sm}\bar{Q}_{i2}u_{jsm} + 2u'_{ism}\bar{Q}_{i3}u_{jsm}$$

$$+ u'_{ism}R_{ii}u_{ism} + u'_{jsm}\tilde{R}_{ij}u_{jsm}]dt \qquad ; i,j = 1,2, \quad i \neq j, \tag{3.49}$$

where

$$\tilde{A}_0 = A_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}A_{21}$$

$$\tilde{B}_{0i} = B_{1i} - \hat{A}_{12}\hat{A}_{22}^{-1}B_{2i}$$

$$\tilde{Q}_{i1} = Q_{i1} - Q_{i2}\hat{A}_{22}^{-1}A_{21} - (\hat{A}_{22}^{-1}A_{21})'Q'_{i2} + (\hat{A}_{22}^{-1}A_{21})'\tilde{Q}_{i3}\hat{A}_{22}^{-1}A_{21}$$

$$\tilde{Q}_{i2} = -Q_{i2}\hat{A}_{22}^{-1}B_{2i} + (\hat{A}_{22}^{-1}A_{21})'\tilde{Q}_{i3}\hat{A}_{22}^{-1}B_{2i} + (\hat{A}_{22}^{-1}A_{21})'K_{if}B_{2i}$$

$$\bar{Q}_{i2} = -Q_{i2}\hat{A}_{22}^{-1}B_{2j} + (\hat{A}_{22}^{-1}A_{21})'\tilde{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + (\hat{A}_{22}^{-1}A_{21})'K_{jf}B_{2j}R_{jj}^{-1}R_{ij}$$

$$\bar{Q}_{i3} = B'_{2i}(\hat{A}_{22}^{-1})'\tilde{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + B'_{2i}K_{if}\hat{A}_{22}^{-1}B_{2j} + B'_{2i}(\hat{A}_{22}^{-1})'K_{jf}B_{2j}R_{jj}^{-1}R_{ij}$$

$$\tilde{R}_{ij} = R_{ij} + B'_{2j}(\hat{A}_{22}^{-1})'\tilde{Q}_{i3}\hat{A}_{22}^{-1}B_{2j} + B'_{2j}(\hat{A}_{22}^{-1})'K_{jf}B_{2j}R_{jj}^{-1}R_{ij} + R_{ij}R_{jj}^{-1}B'_{2j}K_{jf}\hat{A}_{22}^{-1}B_{2j}$$

and we assume that $\begin{bmatrix} R_{11} & \bar{Q}_{13} \\ \bar{Q}_{23} & R_{22} \end{bmatrix}$ is nonsingular.

We will show in this section that the reduction process we have described leads to a well-posed reduced game. Note that the modified slow subsystem and performance indices are of the same form as in the slow problem considered in Section 3.2. However, the system matrices and performance coefficients contain information about the fast low order game. Examining equation (3.44) we see that we still have a control composed of fast and slow parts. However, since we substitute the explicit form for the fast part into the state equation and performance indices before the slow state equation and performance indices are formed, the slow modes are dependent on the fast modes. In Sections 3.2 and 3.3 the fast modes and slow modes were completely separated.

The closed-loop Nash strategy for (3.49) subject to (3.48) is given by

$$u_{ism} = -R_{ii}^{-1}[\tilde{Q}_{i2}'x_{sm} + \tilde{B}_{0i}'K_{ism}x_{sm} + \bar{Q}_{i3}u_{jsm}] \tag{3.50}$$

$$= -\bar{M}_{is}x_{sm} \tag{3.51}$$

where K_{1sm}, K_{2sm} satisfy the coupled Riccati equations

$$0 = -\tilde{Q}_{i1} - K_{ism}\tilde{A}_0 - \tilde{A}_0'K_{ism} + [K_{ism}\tilde{B}_{0j} + \bar{Q}_{i2}]\bar{M}_{js} + \bar{M}_{js}'[\tilde{B}_{0j}'K_{ism} + \bar{Q}_{i2}'] - \bar{M}_{js}'\tilde{R}_{ij}\bar{M}_{js}$$
$$+ \bar{M}_{is}'R_{ii}\bar{M}_{is} \qquad ; \; i,j = 1,2; \quad i \neq j. \tag{3.52}$$

Of course (3.51) and (3.42) are only subsystem optimal. That is, as they stand we cannot apply them to the original system (3.1). Following the methodology of [16] we form a "composite" control involving both fast and slow control coefficients. The form for the composite control is suggested by (3.44). Forming

$$u_{ic} = -\bar{M}_{is}x_1 - R_{ii}^{-1}B_{2i}'K_{if}x_2 \tag{3.53a}$$

$$= -R_{ii}^{-1}B_i' \begin{bmatrix} K_{ism} & 0 \\ \mu K_{im}' & \mu K_{if} \end{bmatrix} x \tag{3.53b}$$

$$= -R_{ii}^{-1}B_i'M_{ic}x \tag{3.53c}$$

where

$$K_{im} = \{-Q_{i2} - A_{21}'K_{if} - K_{ism}\hat{A}_{12} + (K_{jsm}\tilde{B}_{0j} + \tilde{Q}_{j2} - [K_{ism}\tilde{B}_{0i} + \tilde{Q}_{i2}]R_{ii}^{-1}\bar{Q}_{j3}')\cdot$$
$$\cdot[R_{jj} - \bar{Q}_{i3}'R_{ii}^{-1}\bar{Q}_{j3}']^{-1}[B_{2j}'K_{if} - R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\}\hat{A}_{22}^{-1}. \tag{3.54}$$

We note that the coefficient of x_1 involves both fast and slow Riccati gains while the coefficient of x_2 involves only fast Riccati gains.

If the composite control (3.53) is applied to (3.1) for performance indices (3.2) a suboptimal performance cost results which can be written as

$$J_{ic} = \frac{1}{2}\, x_0' P_{ic} x_0 \tag{3.55}$$

where

$$\begin{aligned} 0 = {} & P_{ic}[A - B_i R_{ii}^{-1} B_i' M_{ic} - B_j R_{jj}^{-1} B_j' M_{jc}] + [A - B_i R_{ii}^{-1} B_i' M_{ic} - B_j R_{jj}^{-1} B_j' M_{jc}]' P_{ic} + Q_i \\ & + M_{ic}' B_i R_{ii}^{-1} B_i' M_{ic} + M_{jc}' B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' M_{jc}. \end{aligned} \tag{3.56}$$

To compare the optimal performance cost (3.13) and the composite performance cost (3.55) we need the following conditions:

<u>Condition b</u>

$$[\mathcal{A}_3 - \mathcal{C}_1 \mathcal{A}_3^{-1} \mathcal{C}_2] \tag{3.57}$$

is nonsingular where

$$\mathcal{A}_3 = I \otimes \hat{A}_{22}' + \hat{A}_{22}' \otimes I$$

$$\mathcal{C}_i = I \otimes C_i' + C_i' \otimes I$$

and

$$C_i = B_{2i} R_{ii}^{-1} B_{2i}' K_{jf} - B_{2i} R_{ii}^{-1} R_{ji} R_{ii}^{-1} B_{2i}' K_{if}.$$

<u>Condition c</u>

$$\begin{bmatrix} \check{\mathcal{A}}_1 & \check{\mathcal{B}}_1 \\ \check{\mathcal{B}}_2 & \check{\mathcal{A}}_2 \end{bmatrix} \tag{3.58}$$

is nonsingular where

$$\check{\mathcal{A}}_i = I \otimes \check{A}_i' + \check{A}_i' \otimes I$$

$$\check{\mathcal{B}}_i = -I \otimes \check{B}_i' - \check{B}_i' \otimes I$$

and

$$\check{A}_i = \tilde{A}_0 - \tilde{B}_{0i} \bar{M}_{is} - \tilde{B}_{0j} \bar{M}_{js} + N_{i1} B_{2j} R_{jj}^{-1} \tau_i$$

$$\check{B}_i = [B_{1j} + N_{i2} B_{2j}] R_{jj}^{-1} \tau_i$$

$$\tau_i = B_{1j}' K_{ism} + B_{2j}' K_{im}' - R_{ij} \bar{M}_{js} + [B_{2j}' K_{if} - R_{ij} R_{jj}^{-1} B_{2j}' K_{jf}] \hat{A}_{22}^{-1} [-A_{21} + B_{21} \bar{M}_{is} + B_{2j} \bar{M}_{js}]$$

$$N_{i1} = \tilde{B}_{0i}[R_{ii}^{-1}B_{2i}'K_{jf} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}T_i^{-1}$$

$$N_{i2} = \{-\hat{A}_{12}\hat{A}_{22}^{-1} + B_{ij}[R_{jj}^{-1}B_{2j}K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i}[R_{ii}^{-1}B_{2i}'K_{jf} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}\}T_i^{-1}$$

$$T_i = I - B_{2j}[R_{jj}^{-1}B_{2j}'K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i}[R_{ii}^{-1}B_{2i}'K_{jf} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}.$$

If these conditions hold we have the following theorem.

Theorem 3.5: If

1) the fast game (3.41a), (3.41b) has a unique stabilizing closed-loop Nash solution,

2) the modified slow game (3.48), (3.49) has a unique stabilizing closed-loop Nash solution,

3) Condition b is satisfied,

and 4) Condition c is satisfied,

then K_i, the solution of (3.4), possesses a power series expansion at $\mu = 0$, that is,

$$K_i(\mu) = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} K_{i1}^{(j)} & \mu K_{i2}^{(j)} \\ \mu K_{i2}^{'(j)} & \mu K_{i3}^{(j)} \end{bmatrix} \tag{3.59}$$

$$K_{ik}^{(j)} = \left.\frac{\partial^j K_{ik}(\mu)}{\partial \mu^j}\right|_{\mu=0} \quad ; \; i = 1,2; \quad k = 1,2,3. \tag{3.60}$$

Furthermore, the matrices $K_{i1}^{(0)}$, $K_{i2}^{(0)}$, and $K_{i3}^{(0)}$ satisfy the identities

$$K_{i1}^{(0)} = K_{ism}$$

$$K_{i2}^{(0)} = K_{im}$$

$$K_{i3}^{(0)} = K_{if}.$$

Proof: The proof is given in Appendix I. While the proof is similar to the proof of Theorem 3.1 the complexity warrants inclusion.

An immediate result of Theorem 3.4 is that

$$u_{ic} = u_i + 0(\mu) \tag{3.61}$$

where u_i is the optimal Nash control for (3.1), (3.2). This can be shown easily by substituting (3.59) into (3.3) and letting $\mu = 0$. The identities found in Theorem 3.4 yield (3.61). Furthermore we have the following results.

Theorem 3.6: If the fast game (3.41a), (3.41b) has a unique stabilizing closed-loop Nash solution and the modified slow game (3.48), (3.49) has a unique stabilizing closed-loop Nash solution, then P_{ic} possesses a power series expansion at $\mu = 0$, that is,

$$P_{ic} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} P_{ic1}^{(j)} & \mu P_{ic2}^{(j)} \\ \mu P_{ic2}^{'(j)} & \mu P_{ic3}^{(j)} \end{bmatrix}. \tag{3.62}$$

Proof: The proof is similar to the proof of Theorem 3.2 and is omitted for brevity.

As a result of K_i and P_{ic} possessing power series expansions at $\mu = 0$ it is easy to show that their difference also has a power series expansion at $\mu = 0$. Comparison of the optimal Nash performance with the composite performance costs gives the following theorem.

Theorem 3.7: The first terms of the power series expansion at $\mu = 0$ of K_i and P_{ic} are the same, that is,

$$J_{ic} = J_i + 0(\mu). \tag{3.63}$$

Proof: The proof is similar to the proof of Theorem 3.3 and is omitted for brevity.

Theorem 3.6 shows the closeness of the performance indices when an approximation u_{ic} (3.53) of the Nash controls are used instead of the Nash controls (3.3). Similar to Theorem 3.4 we establish that (u_{1c},u_{2c}) satisfies an asymptotic Nash relationship.

Theorem 3.8: The composite control pair (u_{1c},u_{2c}) satisfies the asymptotic Nash relationships

$$J_1(u_{1c},u_{2c}) \leq J_1(u_1,u_{2c}) + 0(\mu) \tag{3.64a}$$

$$J_2(u_{1c},u_{2c}) \leq J_2(u_{1c},u_2) + 0(\mu) \tag{3.64b}$$

where u_i are in some allowed strategy set.

Proof: The proof is similar to the proof of Theorem 2.4 and is omitted for brevity.

Thus far we have not changed the structure of the controller for the full order system (3.1). That is, the composite controls are a function of both x_1 and x_2 as are the optimal Nash closed-loop controls. If it is desired to implement the control as a function of x_1 only to achieve an $0(\mu)$ approximation of the optimal cost we use the following procedure. Substitute the slow control (3.51) into (3.47). This gives as an approximation of x_2

$$\tilde{x}_2 = -\hat{A}_{22}^{-1}[A_{21}-B_{21}\overline{M}_{1s}-B_{22}\overline{M}_{2s}]x_1. \tag{3.65}$$

If (3.65) is substituted for x_2 in the composite control we have a "lower order" control as a function of x_1 only. This lower order control is

$$u_{i\ell} = -\{\overline{M}_{is}-R_{ii}^{-1}B_{2i}'K_{if}\hat{A}_{22}^{-1}[A_{21}-B_{21}\overline{M}_{1s}-B_{22}\overline{M}_{2s}]\}x_1 \tag{3.66a}$$

$$= -R_{ii}^{-1}B_i'M_{i\ell}x \tag{3.66b}$$

where

$$M_{i\ell} = \begin{bmatrix} K_{ism} & 0 \\ \mu\tilde{K}'_{im} & 0 \end{bmatrix}$$

and

$$\tilde{K}_{im} = K_{im} - [A_{21} - B_{21}\overline{M}_{1s} - B_{22}\overline{M}_{2s}]'(\hat{A}_{22}^{-1})'K_{if}.$$

If (3.66) is applied to the full order system (3.1) for performance indices (3.2) a cost results which can be written as

$$J_{i\ell} = \tfrac{1}{2}x_0' P_{i\ell} x_0 \tag{3.67}$$

where $P_{i\ell}$ is the positive semidefinite solution of the Lyapunov equation

$$\begin{aligned} 0 = P_{i\ell}[A - B_i R_{ii}^{-1} B_i' M_{i\ell} - B_j R_{jj}^{-1} B_j' M_{j\ell}] + [A - B_i R_{ii}^{-1} B_i' M_{i\ell} - B_j R_{jj}^{-1} B_j' M_{j\ell}]' P_{i\ell} + Q_i \\ + M_{i\ell}' B_i R_{ii}^{-1} B_i' M_{i\ell} + M_{j\ell}' B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' M_{j\ell}. \end{aligned} \tag{3.68}$$

Following the method used earlier in this chapter we have the following theorem.

<u>Theorem 3.9</u>: If A_{22} is stable and the modified slow game (3.48), (3.49) has a unique stabilizing closed-loop Nash solution, then $P_{i\ell}$ possesses a power series expansion at $\mu = 0$, that is,

$$P_{i\ell} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} P_{i\ell 1}^{(j)} & \mu P_{i\ell 2}^{(j)} \\ \mu P_{i\ell 2}^{\prime(j)} & \mu P_{i\ell 3}^{(j)} \end{bmatrix}. \tag{3.69}$$

<u>Proof</u>: The proof is similar to the proof of Theorem 3.2 and is omitted for brevity.

Since $P_{i\ell}$ and P_{ic} possess power series expansions at $\mu = 0$ it can be shown that their difference also has a power series expansion at $\mu = 0$.

Comparison of the composite performance costs and the lower order performance costs gives the following theorem.

Theorem 3.10: The first terms of the power series expansion at $\mu = 0$ of $P_{i\ell}$ and P_{ic} are the same, that is

$$J_{i\ell} = J_{ic} + O(\mu). \tag{3.70}$$

Proof: The proof is similar to the proof of Theorem 3.3 and is omitted for brevity.

As a result of Theorem 3.10, it can be seen that the costs at $\mu = 0$ for the full order optimal Nash game, the full order game with composite control applied, and the full order game with the lower order control applied are the same. Thus we have shown that the modified slow game which leads to the composite control u_{ic} in (3.53) and to the reduced control $u_{i\ell}$ in (3.66) is a well-posed reduced order closed-loop Nash game, without having to modify the original quadratic cost functions in (3.2).

We would now like to examine the relationship between the state trajectories of the full order system (3.1) when the composite controls are applied and the state trajectories of the fast and modified slow subsystems when the fast and modified slow controls are respectively applied.

Theorem 3.11: If the controls

$$u_{ism} = -\bar{M}_{is} x_{sm} \tag{3.71}$$

$$u_{if} = -R_{ii}^{-1} B_{2i}' K_{if} x_f \tag{3.72}$$

$$u_{ic} = -\bar{M}_{is} x_1 - R_{ii}^{-1} B_{2i}' K_{if} x_2 \tag{3.73}$$

are applied to systems (3.48), (3.40), and (3.1) respectively, and if $[\tilde{A}_0 - \tilde{B}_{01}\bar{M}_{1s} - \tilde{B}_{02}\bar{M}_{2s}]$ is stable then

$$x_1(t) = x_{sm}(t) + O(\mu) \tag{3.74}$$

$$x_2(t) = -\hat{A}_{22}^{-1}\hat{A}_{21}x_{sm}(t) + x_f(t) + O(\mu) \tag{3.75}$$

hold for all finite $t > 0$. If $\hat{A}_{22}$ is also stable then (3.82), (3.83) hold for all $t \in [0,\infty)$.

Proof: See Appendix J.

Thus we have shown that the state trajectories resulting from application of the composite control can be approximated to first order by combinations of trajectories from the fast and modified slow subsystems. It can also be shown that the state trajectories that result when the lower order control is applied to (3.1) are the same as those of (3.74), (3.75), however with the fast state trajectories being the ones that result if no control is applied to the fast subsystem. In fact the slow part of the $x_1(t)$ and $x_2(t)$ trajectories are the same for both the composite and lower order control applications, but the fast part of $x_2(t)$ differs. We cannot give any relationships like (3.40) or (3.64) even though the cost with the lower order control is $O(\mu)$ to the optimal Nash cost because of the boundary layer where the state trajectories are not within $O(\mu)$. This indicates that the lower order control ignores the fast part of the states and this apparently contributes some small amount to the cost which is enough to preclude our saying that $(u_{1\ell}, u_{2\ell})$ is asymptotically Nash. However, we could say that outside some boundary layer the lower order control pair behaves as if it were the asymptotically Nash composite controls.

4. DESIGN EXAMPLE

4.1. Interconnected Power System

Consider the problem of an interconnected power system. The interconnection is assumed to consist of two identical power areas each consisting of one non-reheat steam turbine generator. Together with the tie-line power flow equation the system is modeled as a ninth order system of the form

$$\dot{x} = Ax + B_1u_1 + B_2u_2. \tag{4.1}$$

The model we present is taken from [23,24] and appears in nearly the same form as in [19]. The only difference being that in the tie-line we assume a power angle of 45^o instead of 30^o and a turbine time constant of .3 instead of .2. The system model is for variations from typical values. Each system consists of a power balance equation $(\Delta\dot{f})$, a governor equation $(\Delta\dot{a})$, a non-reheat steam turbine equation $(\Delta\dot{P}_G)$, a tie-line equation $(\Delta\dot{P}_{12})$, and an area control error equation $(\dot{v})$. We assume a constant load disturbance and define all variables to be deviations from their steady state values.

The components of the control vectors are:

$u_1 = \Delta P_{c_1}$ = speed changer variation for turbine one

$u_2 = \Delta P_{c_2}$ = speed changer variation for turbine two.

The components of the state vector are:

$x_1 = v_1 \quad = \int (ACE)_1 dt = \int (\Delta P_{tie_1} + b_{s_1}\Delta f_1)dt$

= integral of the area control error for area 1 where ΔP_{tie_1} is the tie-line power flow variation from area 1 to area 2

$x_2 = \Delta f_1$ = area 1 frequency variation in per unit (p.u.)

$x_3 = \Delta P_{12}$ = tie-line power flow variation from area 1 to area 2 in p.u. (note that $\Delta P_{12} = -\Delta P_{21} = \Delta P_{tie_1}$)

$x_4 = v_2 = \int (ACE)_2 dt = \int (-\Delta P_{tie_1} + b_{s_2} \Delta f_2) dt$

= integral of the area control error for area 2

$x_5 = \Delta f_2$ = area 2 frequency variation in p.u.

$x_6 = \Delta a_1$ = turbine one valve position variation in p.u.

$x_7 = \Delta P_{G_1}$ = turbine one output power variation in p.u.

$x_8 = \Delta a_2$ = turbine two valve position variation in p.u.

$x_9 = \Delta P_{G_2}$ = turbine two output power variation in p.u.

The parameters and typical values are:

T_{t_i} = turbine i time constant = .3

T_{G_i} = governor i time constant = .08

T_i = system i inertia time constant = 20

r_i = speed regulation = .25

Low values of r_i correspond to strong damping and high values to weak damping. The value we have chosen is relatively high to give good separation in the eigenvalues.

$D_i = .5$

$b_{s_i} = D_i + \frac{1}{r_i}$ = bias parameter for area i = 4.5

T_{12} = synchronizing power flow coefficient = 26.7

This assumes that the tie line has an operating capacity of 10% of area capacity (.1 p.u.) and operates at a power angle of 45^o.

The state space model in the form of equation (4.1) is

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \hdashline \dot{x}_6 \\ \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \end{bmatrix}
=
\left[\begin{array}{ccccc:cccc}
0 & b_{s_1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/T_1 & -1/D_1T_1 & 0 & 0 & 0 & 1/D_1T_1 & 0 & 0 \\
0 & T_{12} & 0 & 0 & -T_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & b_{s_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1/D_2T_2 & 0 & -1/T_2 & 0 & 0 & 0 & 1/D_2T_2 \\
\hdashline
0 & -1/r_1T_{G_1} & 0 & 0 & 0 & -1/T_{G_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/T_{t_1} & -1/T_{t_1} & 0 & 0 \\
0 & 0 & 0 & 0 & -1/r_2T_{G_2} & 0 & 0 & -1/T_{G_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/T_{t_2} & -1/T_{t_2}
\end{array}\right] \cdot
$$

$$
\cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \hdashline x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 1/T_{G_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1
+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 1/T_{G_2} \\ 0 \end{bmatrix} u_2 \qquad (4.2)
$$

The states have been written in the particular groupings that they are in in order to put the system into the singular perturbation form presented in equation (3.1). States one through five vary more slowly than states six through nine. That is, we assume that turbine output power and valve position variations change more rapidly than frequency, tie-line power flow, and the integral of the area control error.

The eigenvalues of A are

$$0,\ 0,\ -12.64,\ -12.64,\ -.14\pm j2.43,\ -2.69,\ -2.69,\ -.55. \qquad (4.3)$$

We clearly have two groupings, namely the slow eigenvalues

$$0,\ 0,\ -.14\pm j2.43,\ -.55 \qquad (4.4)$$

and the fast eigenvalues

$$-12.64,\ -12.64,\ -2.69,\ -2.69. \qquad (4.5)$$

We take the spread of these eigenvalues as our parameter μ. For ease of computation

$$\mu = .01. \qquad (4.6)$$

To check that the system is in the correct form, the eigenvalues of the lower right hand 4×4 block of A should approximate the fast eigenvalues. The eigenvalues of A_{22}/μ are

$$-3.33,\ -3.33,\ -12.50,\ -12.50. \qquad (4.7)$$

Clearly the eigenvalues are close so we write A in its block form corresponding to system (3.1)

$$A_{11} = \begin{bmatrix} 0 & 4.5 & 1 & 0 & 0 \\ 0 & -0.5 & -.1 & 0 & 0 \\ 0 & 26.7 & 0 & 0 & -26.7 \\ 0 & 0 & -1 & 0 & 4.5 \\ 0 & 0 & .1 & 0 & -.05 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & .1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & -.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -.125 & 0 & 0 & 0 \\ .0333 & -.0333 & 0 & 0 \\ 0 & 0 & -.125 & 0 \\ 0 & 0 & .0333 & -.0333 \end{bmatrix}$$

$$B_{11} = B_{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} .125 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\ 0 \\ .125 \\ 0 \end{bmatrix}.$$

Since we now have the system representation in the singularly perturbed form we desire, the next step is to formulate performance indices for each player. We choose $Q_i = I_{9\times 9}$ and $R_{ii} = 20$, $R_{ij} = 0$, $i \neq j$. Thus we desire to force the states to zero but we are more concerned with keeping the control energies small.

The circumstances which warrant use of the Nash equilibrium strategy in the finding of controls for the interconnected power system (4.2) occur when cooperation between the decision makers in the two areas cannot be guaranteed. It is also necessary that the decision makers insist on choosing their own control strategy based on minimizing their own performance index rather than letting some higher authority perform the design. However, each control design is affected by the other area's control. If one control is fixed then the other decision makers control is designed optimum with respect to that control. If each control is optimum with respect to the other decision makers control, then the control pair satisfies the Nash equilibrium strategy concept. This mutual optimization can arise naturally in the following manner. If one decision maker announces a strategy, the best that the other decision maker can do is to optimize his performance index with respect to that given strategy. The original decision

maker may then decide to recompute his strategy based on the computed strategy for the second decision maker. Again, the best he can do is to optimize with respect to the announced strategy of the second player. This process then continues until convergence [4] and the convergent control pair satisfies the Nash equilibrium solution concept. It should also be pointed out that this convergence process can be performed prior to the start of the optimization period or at the beginning of the optimization period. In the second case the problem is Nash beginning at the time the control strategies converge rather than from the initial time for the problem.

4.2. Fast and Modified Slow Subsystems

The fast subsystem of (4.2) is of the form in (3.40), (3.41) when the system matrices are as defined in the previous section. The fast controls are given by

$$u_{1f} = [-.03 \quad -.019 \quad 0 \quad 0\;]x_f \tag{4.8}$$

$$u_{2f} = [\;0 \quad 0 \quad -.03 \quad -.019]x_f. \tag{4.9}$$

The gain matrices of the modified slow subsystem are

$$\tilde{A}_0 = \begin{bmatrix} 0 & 4.5 & 1 & 0 & 0 \\ 0 & -.431 & -.1 & 0 & 0 \\ 0 & 26.7 & 0 & 0 & -26.7 \\ 0 & 0 & -1 & 0 & 4.5 \\ 0 & 0 & .1 & 0 & -.431 \end{bmatrix}, \quad \tilde{B}_{01} = \begin{bmatrix} 0 \\ .095 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_{02} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ .095 \end{bmatrix}.$$

The eigenvalues of $\tilde{A}_0$ are

$$0,\ 0,\ -.22 \pm j2.30,\ -.43$$

which are close to the slow eigenvalues of the original open loop system. The controls for the modified slow game are

$$u_{1sm} = [-.22 \quad -2.84 \quad -.026 \quad -.0021 \quad .92]x_{sm} \tag{4.10}$$

$$u_{2sm} = [-.0021 \quad .92 \quad .026 \quad -.22 \quad -2.84]x_{sm}. \tag{4.11}$$

4.3. The Composite and Lower Order Controls

The composite controls are

$$u_{1c} = -[.22 \quad 2.84 \quad .026 \quad .0021 \quad -.92 \,\vdots\, .03 \quad .019 \quad 0 \quad 0\,]x \tag{4.12}$$

$$u_{2c} = -[.0021 \quad -.92 \quad -.026 \quad .22 \quad 2.84 \,\vdots\, 0 \quad 0 \quad .03 \quad .019]x \tag{4.13}$$

and the lower order controls are

$$u_{1\ell} = -[.213 \quad 2.52 \quad .024 \quad .002 \quad -.87 \,\vdots\, 0 \quad 0 \quad 0 \quad 0]x \tag{4.14}$$

$$u_{2\ell} = -[.002 \quad -.87 \quad -.024 \quad .213 \quad 2.52 \,\vdots\, 0 \quad 0 \quad 0 \quad 0]x. \tag{4.15}$$

It is clear from a comparison of the lower order controls and the composite controls that they are close in the first five, or slow state, positions and that the lower order controls completely ignore the fast states of the system. It is desired to compare the composite controls with the optimal Nash controls. The optimal Nash controls are

$$u_{1\ell} = -[.224 \quad 2.88 \quad -.037 \quad .0017 \quad -.85 \,\vdots\, .05 \quad .106 \quad -.0062 \quad -.025]x \tag{4.16}$$

$$u_{2\ell} = -[.0017 \quad -.85 \quad .037 \quad .224 \quad 2.88 \,\vdots\, -.0062 \quad -.025 \quad .05 \quad .106]x. \tag{4.17}$$

Comparing the composite controls and the optimal Nash controls we see that most entries are close, however the third entry corresponding to the tie-line power flow variation has an opposite sign which is unexpected.

To evaluate the performance indices when composite, lower order, and optimal Nash controls are applied to (4.2) we assume that the initial state conditions are zero mean independent random variables with covariance

$$E[x_o x_o'] = 10^{-4} \mathrm{diag}(1,.01,1,1,.01,1,1,1,1). \tag{4.18}$$

These are typical values taken from [24]. It should be noted that we assume that the frequency variations are much smaller than other state variations.

The expected value of the optimal Nash cost given by (3.13) when (4.16) and (4.17) are applied to (4.2) is

$$E\{J_1\} = E\{J_2\} = 13.32 \times 10^{-4}. \tag{4.19}$$

The expected value of the performance indices when the composite controls (4.12) and (4.13) are applied to (4.2) are found from (3.55) to be

$$E\{J_{1c}\} = E\{J_{2c}\} = 13.959 \times 10^{-4}. \tag{4.20}$$

Similarly, the expected value of the performance index given by (3.66) when the lower order controls (4.14), (4.15) are applied to (4.2) is

$$E\{J_{1\ell}\} = 14.075 \times 10^{-4} \tag{4.21a}$$

$$E\{J_{2\ell}\} = 14.0828 \times 10^{-4}. \tag{4.21b}$$

It should be noted that these values probably should be equal and the difference is due to truncation in the computer routine used. However, persistent trials on similar problems yielded the same small but definite difference in the expected values of the costs. Comparing the optimal and composite costs we get

$$\frac{E\{J_{1c}\} - E\{J_1\}}{E\{J_1\}} = \frac{E\{J_{2c}\} - E\{J_2\}}{E\{J_2\}} = .05 \tag{4.22}$$

and see that there is a performance loss of 5 percent. Comparing the optimal and lower order costs we get

$$\frac{E\{J_{1\ell}\} - E\{J_1\}}{E\{J_1\}} = .057 \tag{4.23a}$$

and

$$\frac{E\{J_{2\ell}\} - E\{J_2\}}{E\{J_2\}} = .058 \tag{4.23b}$$

and see that there is a 5.7 and a 5.8 percent loss in performance. It can be seen that there is only about a .8 percent difference between the composite and lower order costs.

In this example we see that the cost resulting when the lower order and composite controls are applied to (4.2) are larger than the optimal Nash cost. However, in general this will not always be true since by both players deviating from the Nash controls it is in general possible to achieve a lowering in the performance costs.

For a comparison of state trajectories resulting when the optimal Nash, composite and lower order controls are applied to (4.2) we first examine the closed-loop eigenvalues for the three cases mentioned. These are presented in Table 4.1. As we can see the eigenvalues from all three cases are relatively close. Of particular interest is that the fast eigenvalues for all three cases are close and that the fast eigenvalues for the composite and lower order cases are nearly identical. Thus, for this example the state trajectories in the boundary layer are nearly the same. This is not the

Table 4.1. Eigenvalues for the closed-loop system when Nash, composite and lower order controls are applied to (4.2)

Nash	Composite	Lower Order
-12.9	-13.03	-12.75
-12.89	-12.98	-12.70
-3.06	-2.65	-2.58
-2.83	-2.50	-2.41
-.44	-.51	-.51
-.27	-.26	-.24
-.21	-.23	-.23
$-.21 \pm j2.44$	$-.17 \pm j2.50$	$-.16 \pm 2.51$

usual case and is due to the fact that the open- and closed-loop eigenvalues of the fast subsystem are very close. Plots of the state trajectories for all three cases are presented in Figures 4.1-4.9 for initial conditions $x(0) = [.01 \;\; .001 \;\; .01 \;\; .01 \;\; .001 \;\; .01 \;\; .01 \;\; .01 \;\; .01]'$. These trajectories confirm the validity of our modified reduction process and show the accuracy which can result from this procedure.

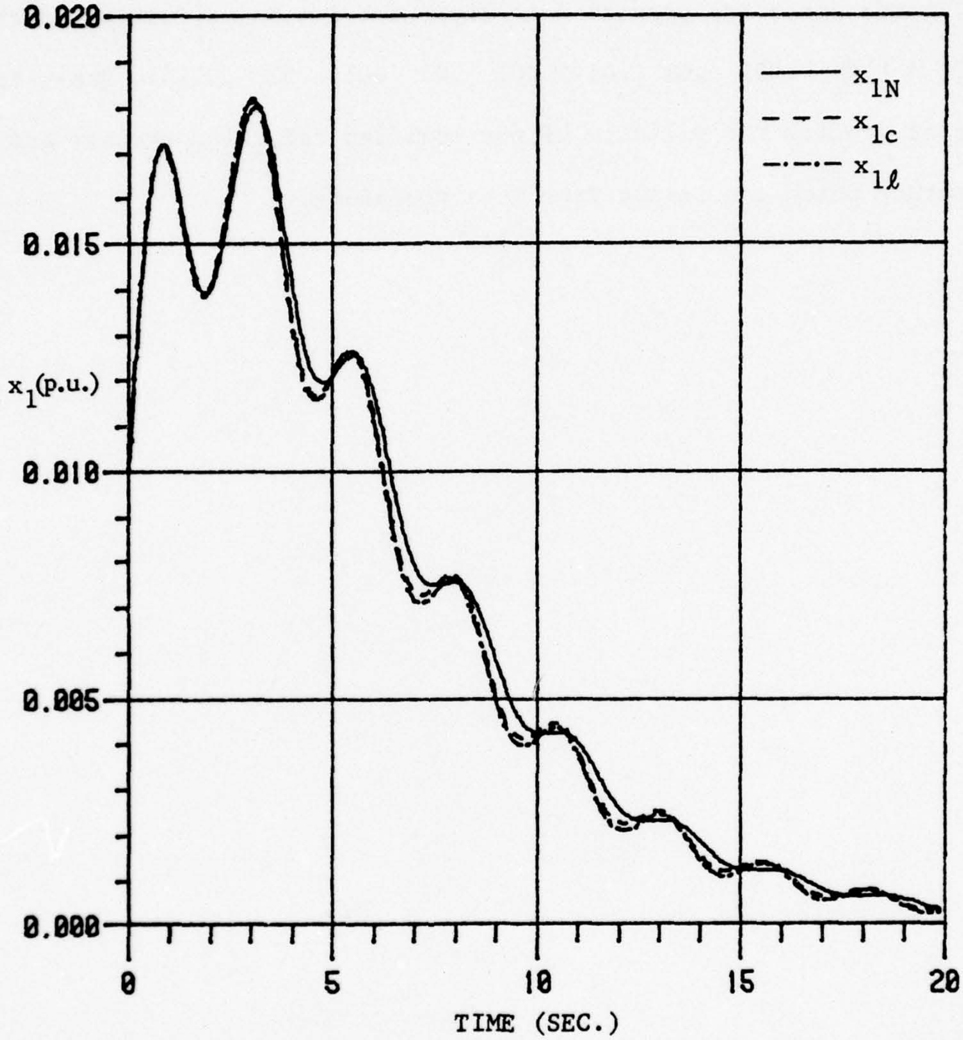

Figure 4.1. Closed-loop state trajectories for the integral of the area control error for area 1 for Nash, composite and lower order control applications.

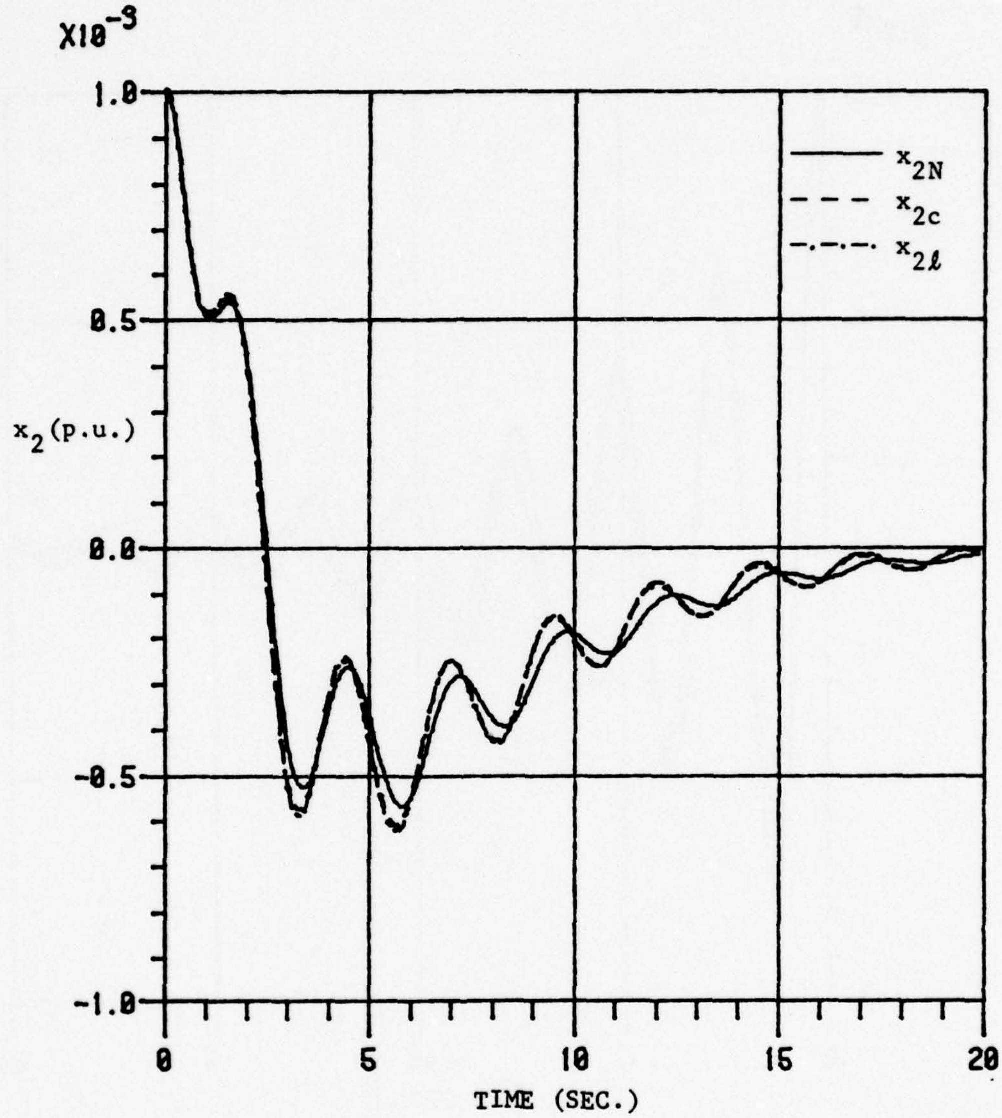

Figure 4.2. Closed-loop state trajectories for area 1 frequency variation for Nash, composite and lower order control application.

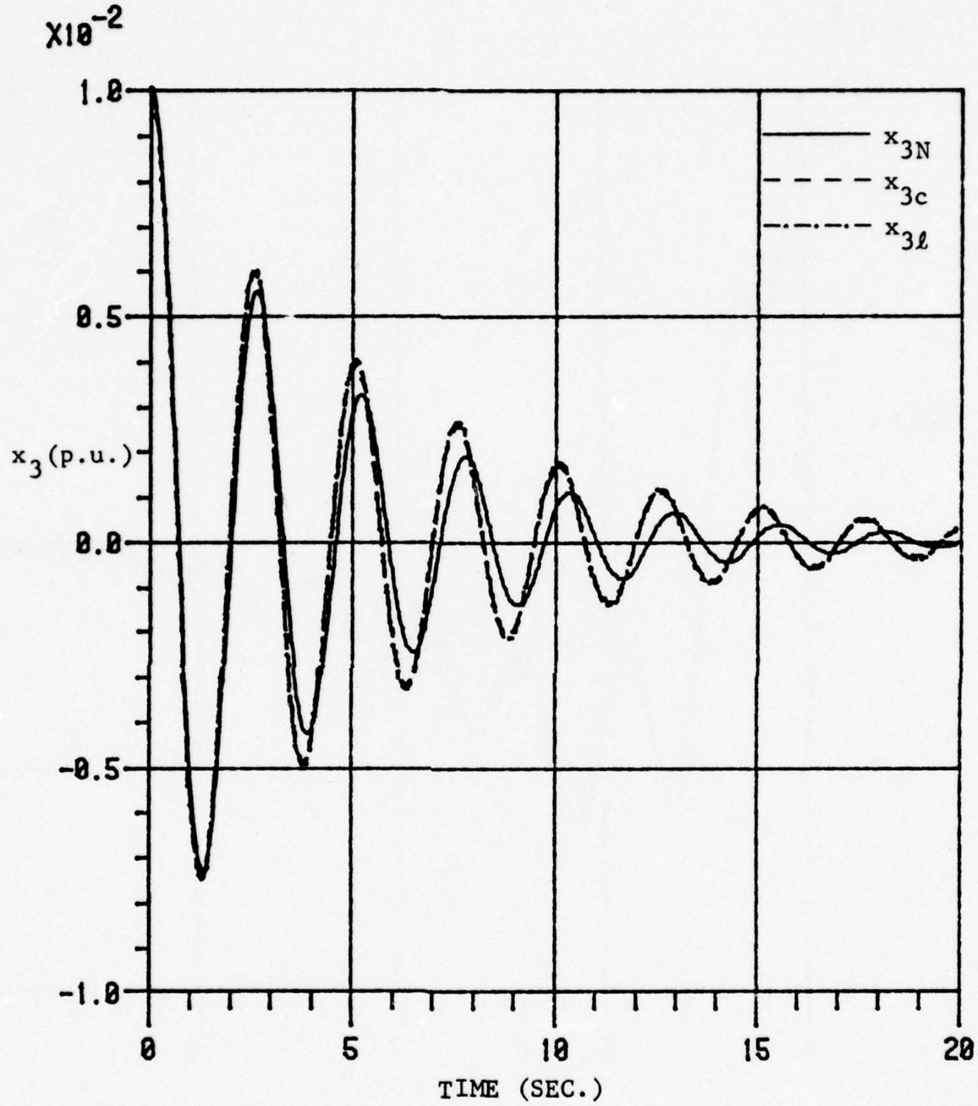

Figure 4.3. Closed-loop state trajectories for tie line power flow variation for Nash, composite and lower order control application.

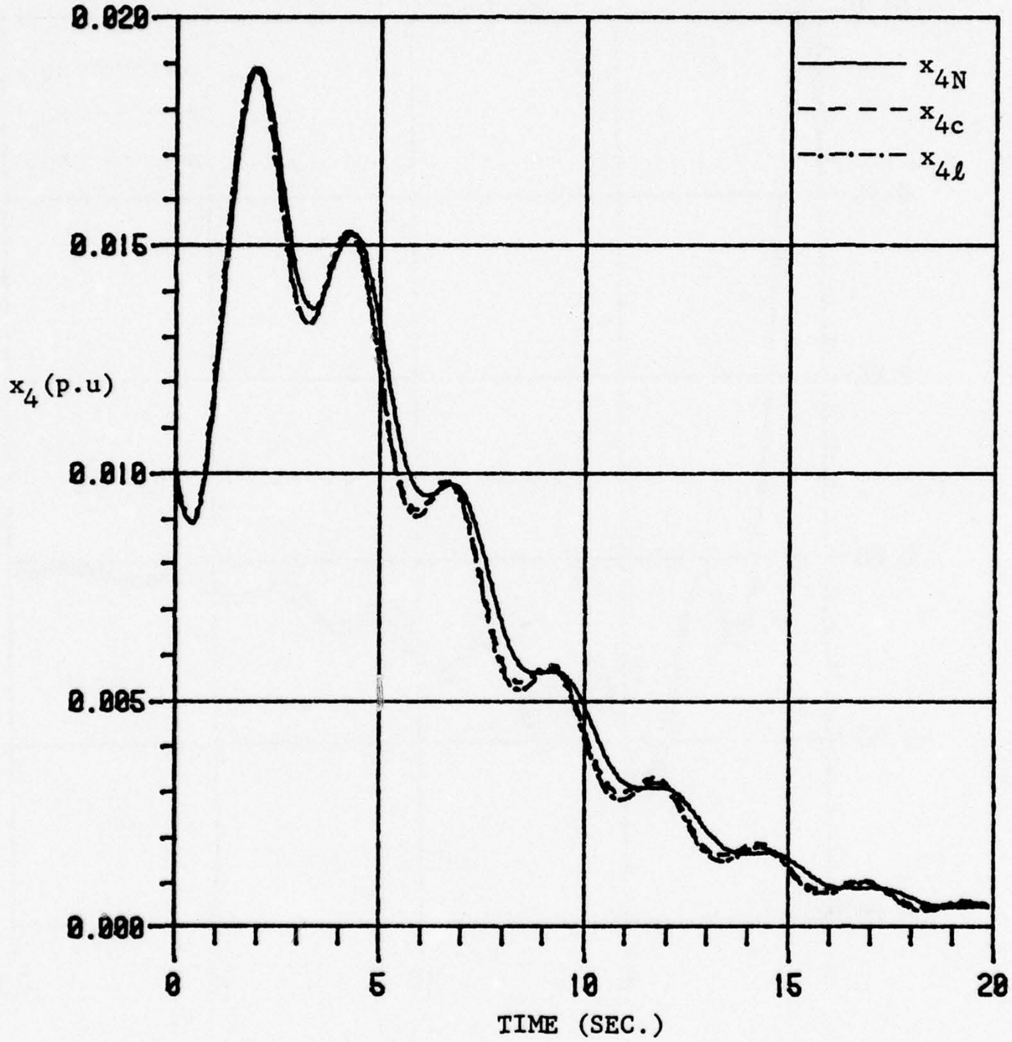

Figure 4.4. Closed-loop state trajectories for the integral of the area control error for area 2 for Nash, composite and lower order control applications.

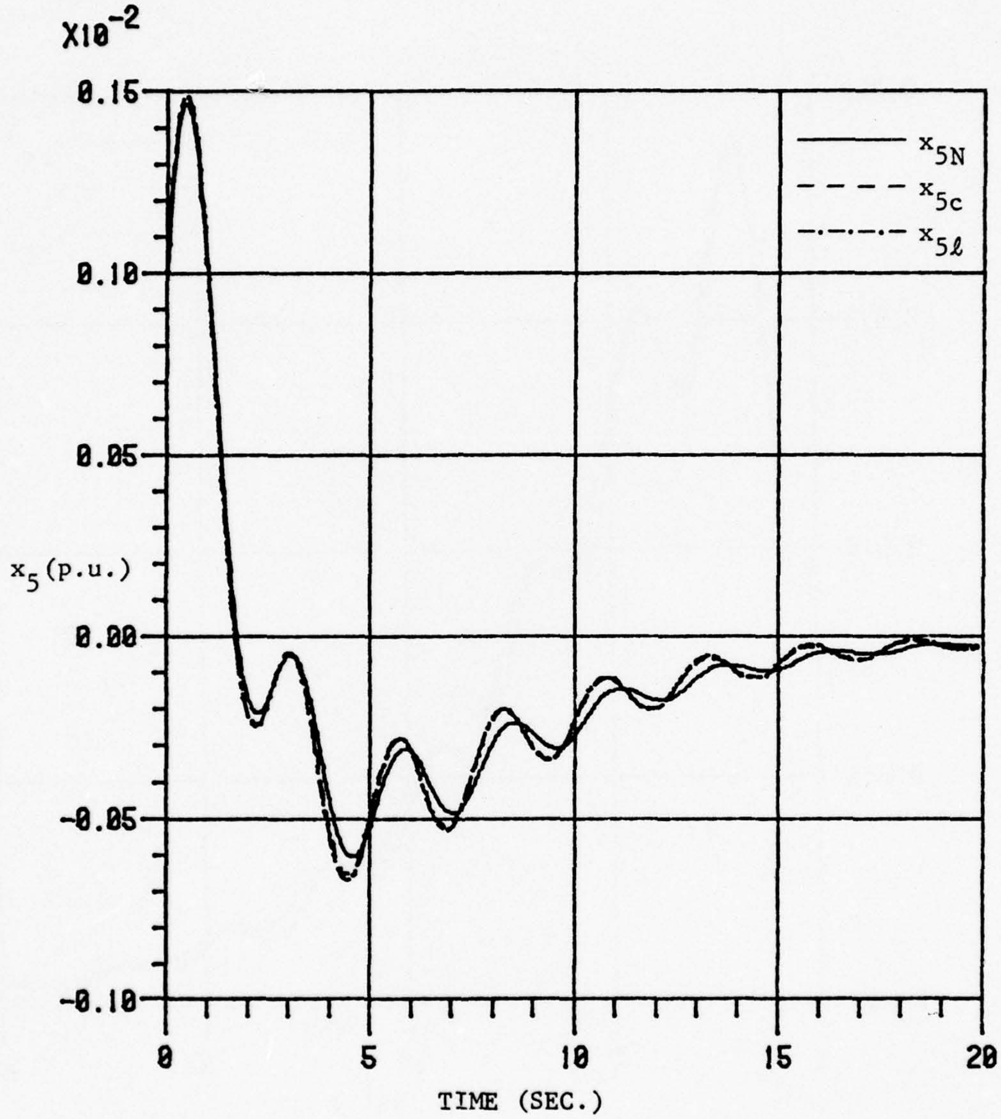

Figure 4.5. Closed-loop state trajectories for area 2 frequency variation for Nash, composite and lower order control application.

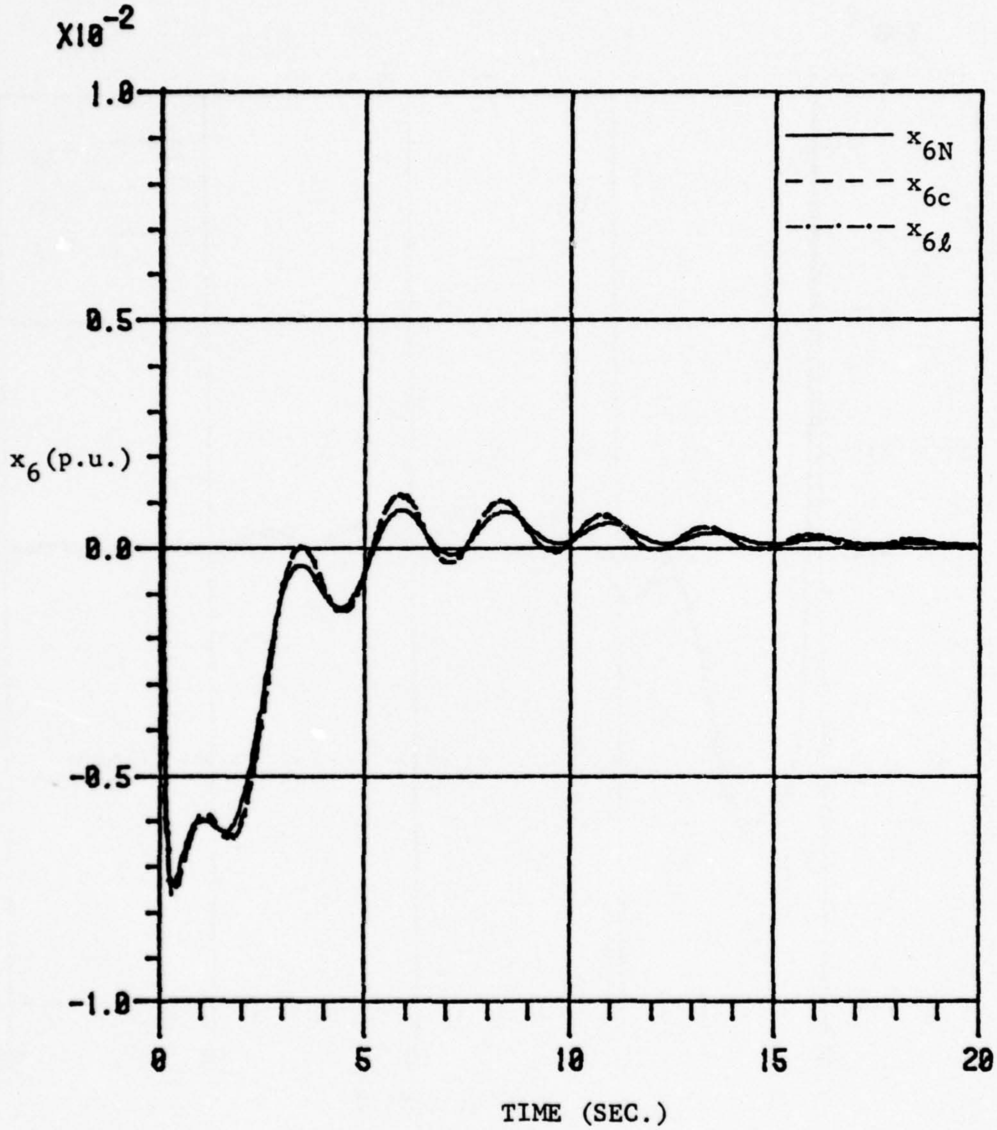

Figure 4.6. Closed-loop state trajectories for turbine one valve position variation for Nash, composite and lower order control application.

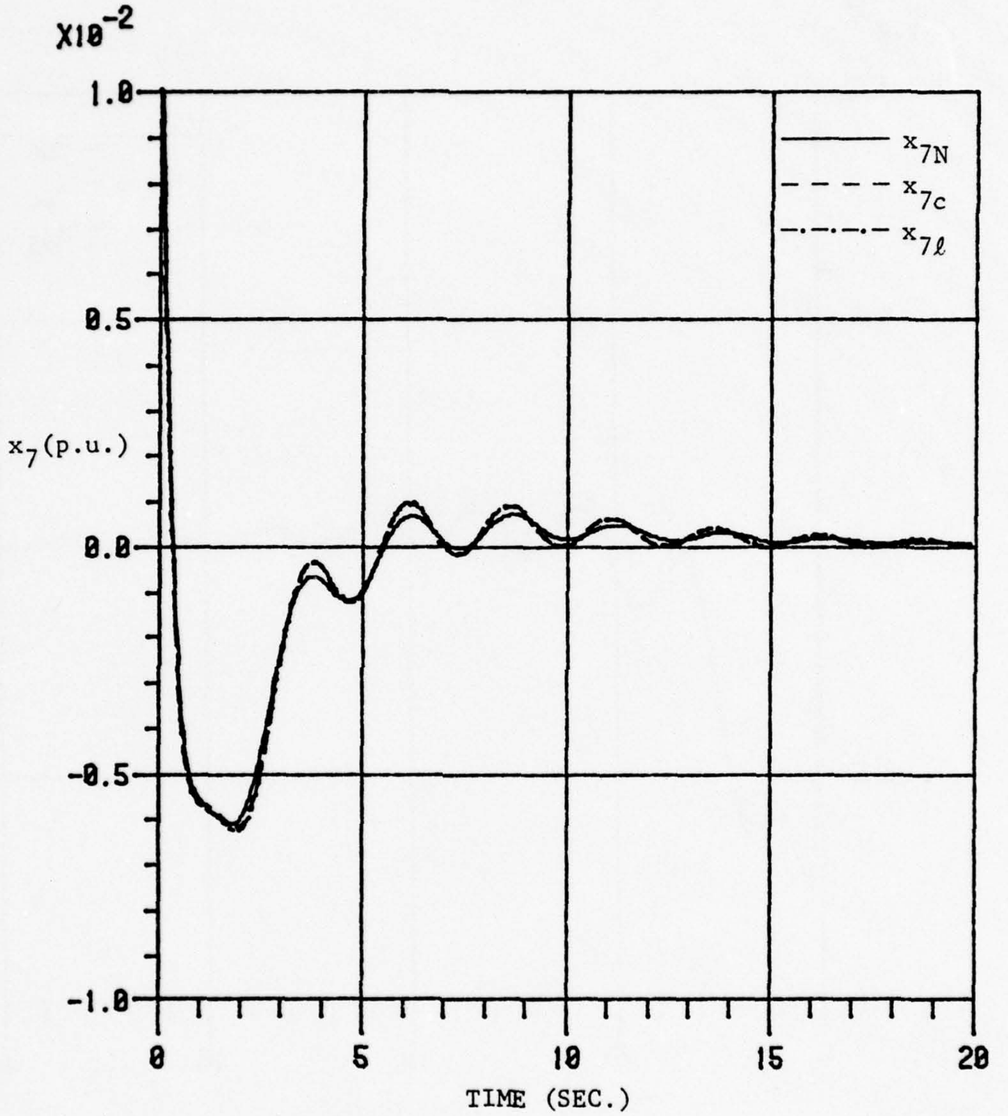

Figure 4.7. Closed-loop state trajectories for turbine one output power variation for Nash, composite and lower order control application.

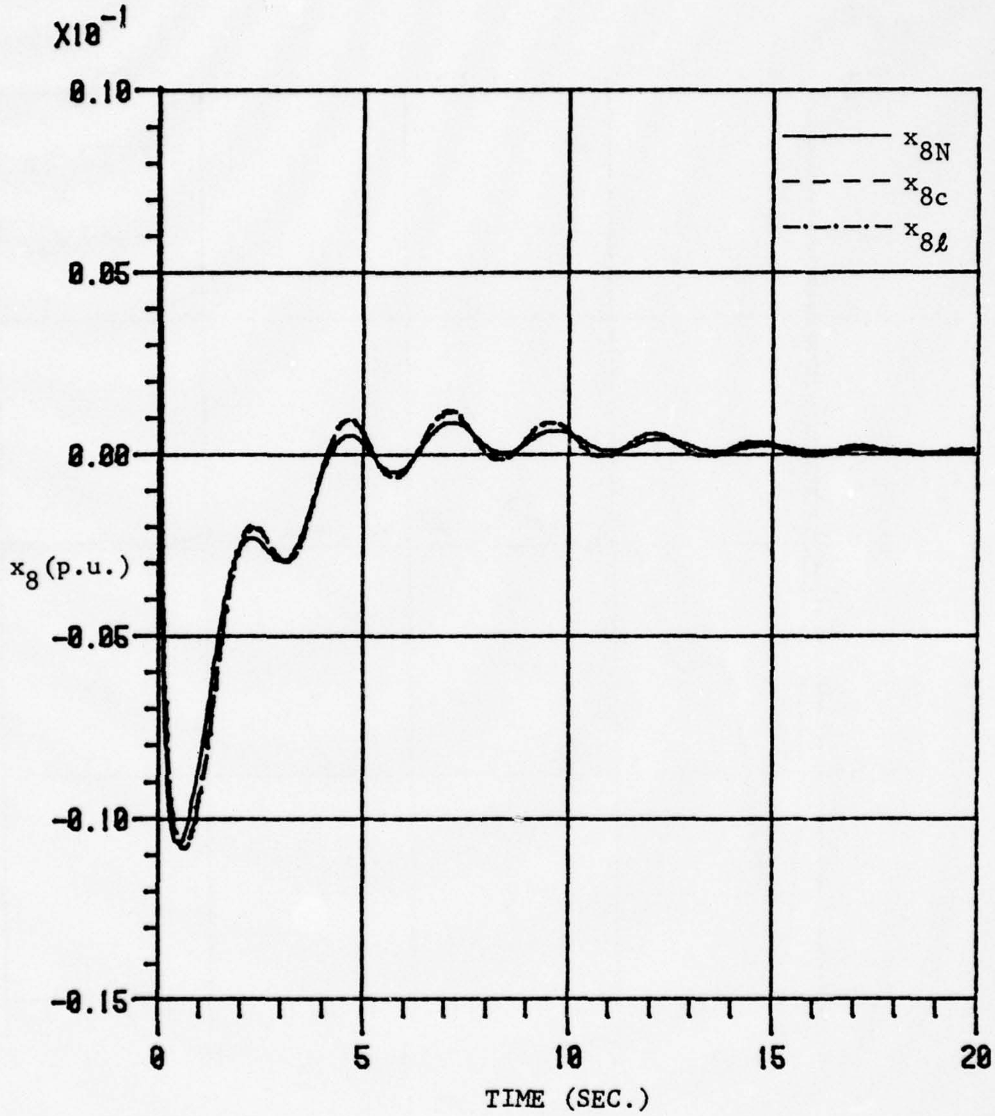

Figure 4.8. Closed-loop state trajectories for turbine two valve position variation for Nash, composite and lower order control application.

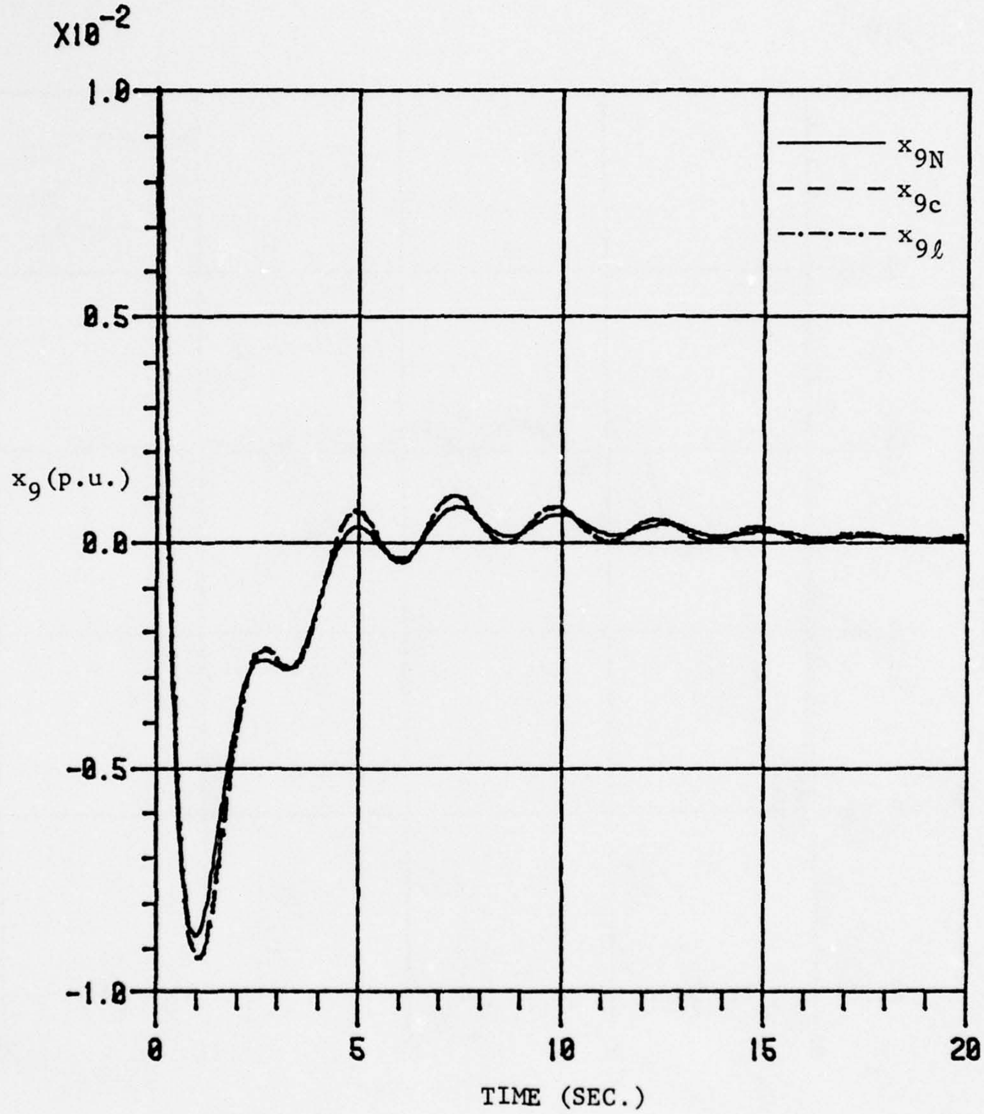

Figure 4.9. Closed-loop state variation for turbine two output power variation for Nash, composite and lower order control application.

5. CONCLUSIONS

This thesis applies the techniques of singular perturbation to the study of both zero-sum and nonzero-sum Nash games for systems with fast and slow modes. Both players are assumed to have the same information about, and model for, the system.

We have shown via example that the usual order reduction procedure for singularly perturbed optimal control systems does not lead to a well-posed problem when extended directly to the linear-quadratic nonzero-sum closed-loop Nash game. If the fast dynamics are not known exactly then only the slow part of the fast states should be incorporated into the performance indices. We have shown that in this case the usual order reduction procedure for singularly perturbed optimal control systems leads to a well-posed problem.

On the other hand, if it is assumed that the fast dynamics are known and are incorporated in both the state equation and performance indices, we have shown that by using a hierarchical reduction procedure developed in Section 3.4 the resulting modified slow game is well-posed. This hierarchical reduction procedure differs from the normal singular perturbation order reduction procedure in that it is a block triangular or sequential process rather than a parallel decomposition. In this sense it is analogous to the reduction method of Kokotovic and Yackel [13] for singularly perturbed optimal control problems where they had the slow Riccati equation dependent on the fast Riccati gain. In our hierarchical decomposition the fast subsystem may be found independently of the slow subsystem but the converse is not true. Also, a choice is provided for

implementing the approximate control as either a function of fast and slow states or as a function of slow states only. As in the optimal control case, knowledge of the value of the small parameter, μ, is not necessary to obtain an $O(\mu)$ feedback control design.

In contrast, for zero-sum Nash games, although the performance indices contain fast modes, the natural order reduction used in optimal control formulations leads to well-posed problems. That is, in zero-sum games it does not matter whether the order reduction is due to ignorance of inadequately modeled fast dynamics or due to computational simplification only.

The near optimal composite controls for the nonzero and zero-sum Nash games tend to the Nash control strategy asymptotically as $\mu \to 0$. A clear advantage of the singular perturbation approach is that the numerical stiffness is alleviated in the approximate control design and the value of the perturbation parameter is not needed in the subsystem calculations.

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APPENDIX A

PROOF OF LEMMA 2.1

Applying (2.18) to (2.7) gives the feedback system

$$\dot{x}_s = (A_o + B_{01}G_{01} + B_{02}G_{02})x_s \quad , \quad x_s(0) = x_{10}. \tag{A.1}$$

Thus

$$x_s(t) = (\exp((A_o + B_{01}G_{01} + B_{02}G_{02})t))x_s(0). \tag{A.2}$$

Similiarly applying (2.19) to (2.11) gives

$$\mu\dot{x}_f = (A_{22} + B_{21}G_{21} + B_{22}G_{22})x_f, \quad x_f(0) = x_{20} - \bar{x}_2(0). \tag{A.3}$$

Hence

$$x_f(t) = (\exp((A_{22} + B_{21}G_{21} + B_{22}G_{22})t/\mu))x_f(0). \tag{A.4}$$

Finally, applying (2.20) to system (2.1) gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{c|c} \begin{matrix} A_{11} + B_{11}(I + G_{21}A_{22}^{-1}B_{21})G_{01} + B_{11}G_{21}A_{22}^{-1}A_{21} + B_{11}G_{21}A_{22}^{-1}B_{22}G_{02} \\ + B_{12}(I + G_{22}A_{22}^{-1}B_{22})G_{02} + B_{12}G_{22}A_{22}^{-1}A_{21} + B_{12}G_{22}A_{22}^{-1}B_{21}G_{01} \end{matrix} & A_{12} + B_{11}G_{21} + B_{12}G_{22} \\ (A_{22} + B_{21}G_{21} + B_{22}G_{22})A_{22}^{-1}(A_{21} + B_{21}G_{01} + B_{22}G_{02})/\mu & (A_{22} + B_{21}G_{21} + B_{22}G_{22})/\mu \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{matrix} x_1(0) = x_{10} \\ x_2(0) = x_{20} \end{matrix} \tag{A.5}$$

Constructing a transformation to diagonalize (A.5) we use the same form as

in [22].

$$T = \left[\begin{array}{c|c} I_1 - \mu HL & -\mu H \\ \hline L & I_2 \end{array}\right], \quad T^{-1} = \left[\begin{array}{c|c} I_1 & \mu H \\ \hline -L & I_2 - \mu LH \end{array}\right] \tag{A.6}$$

where I_1 is an $n_1 x n_1$ identity matrix, I_2 is an $n_2 x n_2$ identity matrix and

$$L = A_{22}^{-1}(A_{21} + B_{21}G_{01} + B_{22}G_{02}) + \mu N \tag{A.7}$$

$$N = (A_{22} + B_{21}G_{21} + B_{22}G_{22})^{-1}A_{22}^{-1}(A_{21} + B_{21}G_{01} + B_{22}G_{02})$$
$$(A_o + B_{01}G_{01} + B_{02}G_{02}) + 0(\mu) \tag{A.8}$$

$$H = (A_{12} + B_{11}G_{21} + B_{12}G_{22})(A_{22} + B_{21}G_{21} + B_{22}G_{22})^{-1} + 0(\mu). \tag{A.9}$$

Neglecting $0(\mu^2)$ terms

$$T\bar{A}T^{-1} = \left[\begin{array}{c|c} \bar{A}_0 & 0 \\ \hline 0 & \bar{A}_2 \end{array}\right] \tag{A.10}$$

where $\bar{A}$ is the system matrix of (A.5) and

$$\bar{A}_0 = A_o + B_{01}G_{01} + B_{02}G_{02} - \mu(A_{12} + B_{11}G_{21} + B_{12}G_{22})N \tag{A.11}$$

$$\bar{A}_2 = (A_{22} + B_{21}G_{21} + B_{22}G_{22})/\mu + (A_{22}^{-1}A_{21} + A_{22}^{-1}B_{21}G_{01}$$
$$+ A_{22}^{-1}B_{22}G_{02} + \mu N)(A_{12} + B_{11}G_{21} + B_{12}G_{22}). \tag{A.12}$$

Thus

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left[\begin{array}{c|c} I_1 & \mu H \\ \hline -L & I_2 - \mu LH \end{array}\right] \left[\begin{array}{c|c} \exp(\bar{A}_0 t) & 0 \\ \hline 0 & \exp(\bar{A}_2 t) \end{array}\right] \left[\begin{array}{c|c} L_1 - \mu HL & -\mu H \\ \hline L & I_2 \end{array}\right] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} I_1 & \mu H \\ \hline -L & I_2 - \mu LH \end{array}\right] \left[\begin{array}{c|c} \exp(\bar{A}_0 t) & 0 \\ \hline 0 & \exp(\bar{A}_2 t) \end{array}\right] \begin{bmatrix} x_s(0) \\ x_f(0) \end{bmatrix} + 0(\mu) \qquad \text{(A.13)}$$

And finally,

$$x_1(t) = (\exp((A_o + B_{01}G_{01} + B_{02}G_{02})t))x_s(0) + 0(\mu) \qquad \text{(A.14)}$$

$$x_2(t) = -A_{22}^{-1}(A_{21} + B_{21}G_{01} + B_{22}G_{02})(\exp((A_o + B_{01}G_{01} + B_{02}G_{02})t))x_s(0)$$

$$+ (\exp((A_{22} + B_{21}G_{21} + B_{22}G_{22})t/\mu))x_f(0) + 0(\mu) \qquad \text{(A.15)}$$

Thus it can be seen that (A.14) and (A.15) together with (A.2) and (A.4) yield (2.21), (2.22) and (2.23). If $[A_{22} + B_{21}G_{21} + B_{22}G_{22}]$ is stable then (2.21) - (2.23) hold for all finite $t > 0$. If in addition $[A_o + B_{01}G_{01} + B_{02}G_{02}]$ is stable then the equations hold for all $t \in [0,\infty)$.

APPENDIX B

PROOF OF THEOREM 2.1

Substitution of (2.25) into (2.4) at $\mu = 0$ gives

$$0 = -Q_1 - A_{11}'S_1 - S_1A_{11} - A_{21}'S_2' - S_2A_{21} + S_1(B_{11}R_1^{-1}B_{11}' + B_{12}R_2^{-1}B_{12}')S_1$$
$$+ S_1(B_{11}R_1^{-1}B_{21}' + B_{12}R_2^{-1}B_{22}')S_2' + S_2(B_{21}R_1^{-1}B_{11}' + B_{22}R_2^{-1}B_{12}')S_1$$
$$+ S_2(B_{21}R_1^{-1}B_{21}' + B_{22}R_2^{-1}B_{22}')S_2' \tag{B.1}$$

$$0 = -Q_2 - A_{21}'S_3 - S_1A_{12} - S_2A_{22} + (S_1B_{11} + S_2B_{21})R_1^{-1}B_{21}'S_3$$
$$+ (S_1B_{12} + S_2B_{22})R_2^{-1}B_{22}'S_3 \tag{B.2}$$

$$0 = -Q_3 - S_3A_{22} - A_{22}'S_3 + S_3(B_{21}R_1^{-1}B_{21}' + B_{22}R_2^{-1}B_{22}')S_3. \tag{B.3}$$

If the fast game (2.11), (2.12) possesses a unique solution then comparison of (B.3) and (2.14) gives

$$S_3 = K_f. \tag{B.4}$$

Let

$$N_1 = S_1B_{11} + S_2B_{21} \tag{B.5}$$

$$N_2 = S_1B_{12} + S_2B_{22}, \tag{B.6}$$

then from (B.2) we have

$$S_2 = (N_1R_1^{-1}B_{21}'S_3 + N_2R_2^{-1}B_{22}'S_3 - Q_2 - A_{21}'S_3 - S_1A_{12})A_{22}^{-1}. \tag{B.7}$$

Defining

$$P_1 = S_1B_{11} + S_2B_{21} - (A_{22}^{-1}A_{21})'S_3B_{21} \tag{B.8}$$

$$P_2 = S_1B_{12} + S_2B_{22} - (A_{22}^{-1}A_{21})'S_3B_{22} \tag{B.9}$$

equation (B.1) reduces to

$$0 = -(\hat{Q}_1 + S_1A_o + A_o'S_1) + P_1R_1^{-1}P_1' + P_2R_2^{-1}P_2'. \tag{B.10}$$

Substituting from (B.3) for S_3 in (B.8) and (B.9) and assuming the appropriate inverse exists

$$[P_2 \vdots P_1] = \begin{bmatrix} B_{22}'\hat{Q}_2' + B_{02}'S_1 \\ B_{21}'\hat{Q}_2' + B_{01}'S_1 \end{bmatrix}' \begin{bmatrix} I-R_2^{-1}B_{22}'S_3A_{22}^{-1}B_{22} & -R_2^{-1}B_{22}'S_3A_{22}^{-1}B_{21} \\ -R_1^{-1}B_{21}'S_3A_{22}^{-1}B_{22} & I-R_1^{-1}B_{21}'S_3A_{22}^{-1}B_{21} \end{bmatrix}^{-1}. \tag{B.11}$$

Substituting (B.11) into (B.10) and using (B.3) gives

$$0 = -(\hat{Q}_1+S_1A_o+A_o'S_1) + \begin{bmatrix} B_{22}'\hat{Q}_2' + B_{02}'S_1 \\ B_{21}'\hat{Q}_2' + B_{01}'S_1 \end{bmatrix}' \begin{bmatrix} \hat{R}_2 & \hat{Q}_3' \\ \hat{Q}_3 & \hat{R}_1 \end{bmatrix}^{-1} \begin{bmatrix} B_{22}'\hat{Q}_2' + B_{02}'S_1 \\ B_{21}'\hat{Q}_2' + B_{01}'S_1 \end{bmatrix}. \tag{B.12}$$

Comparison of (B.12) and (2.10) reveals that if the slow game has a unique stabilizing solution, then

$$S_1 = K_s. \tag{B.13}$$

Finally, substitution of (B.13) and (B.4) into (B.2) gives

$$\begin{aligned} S_2 = {} & [-Q_2-A_{21}'K_f-K_sA_{12} + K_s(B_{11}R_1^{-1}B_{21}'+B_{12}R_2^{-1}B_{22}')K_f]\cdot \\ & \cdot[A_{22}-B_{21}R_1^{-1}B_{21}'K_f-B_{22}R_2^{-1}B_{22}'K_f]^{-1}. \end{aligned} \tag{B.14}$$

APPENDIX C

PROOF OF THEOREM 2.2

The optimal Nash control with $\mu = 0$ is given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} B_{11}'S_1 + B_{21}'S_2' \\ B_{12}'S_1 + B_{22}'S_2' \end{bmatrix} x_1 - \begin{bmatrix} R_1^{-1}B_{21}'S_3 \\ R_2^{-1}B_{22}'S_3 \end{bmatrix} x_2 . \tag{C.1}$$

Replacing S_2 in (C.1) by (B.7) and then replacing S_3 from equation (B.3) gives

$$B_{1i}'S_1 + B_{2i}'S_2' = B_{0i}'S_1 + B_{2i}'\hat{Q}_2' + B_{2i}'S_3A_{22}^{-1}A_{21} + B_{2i}'(A_{22}^{-1})'S_3(B_{21}R_1^{-1}P_1' + B_{22}R_2^{-1}P_2') \qquad i = 1,2. \tag{C.2}$$

Then substituting (B.11) for $[P_1 \vdots P_2]'$ gives

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\left[\begin{array}{c|c} R_1^{-1} & 0 \\ \hline 0 & R_2^{-1} \end{array}\right] \left[\begin{array}{c|c} I - B_{21}'(A_{22}^{-1})'S_3B_{21}R_1^{-1} & -B_{21}'(A_{22}^{-1})'S_3B_{22}R_2^{-1} \\ \hline -B_{22}'(A_{22}^{-1})'S_3B_{21}R_1^{-1} & I - B_{22}'(A_{22}^{-1})'S_3B_{22}R_2^{-1} \end{array}\right] .$$

$$\cdot \left[\begin{array}{c} B_{01}'S_1 + B_{21}'\hat{Q}_2' \\ \hline B_{02}'S_1 + B_{22}'\hat{Q}_2' \end{array}\right] x_1 - \left[\begin{array}{c} R_1^{-1}B_{21}'S_3A_{22}^{-1}A_{21} \\ \hline R_2^{-1}B_{22}'S_3A_{22}^{-1}A_{21} \end{array}\right] x_1 - \left[\begin{array}{c} R_1^{-1}B_{21}'S_3 \\ \hline R_2^{-1}B_{22}'S_3 \end{array}\right] x_2 . \tag{C.3}$$

Straightforward multiplication and then use of equation (2.14) gives

$$\begin{bmatrix} R_1 - B_{21}'K_fA_{22}^{-1}B_{21} & -B_{21}'K_fA_{22}^{-1}B_{22} \\ -B_{22}'K_fA_{22}^{-1}B_{21} & R_2 - B_{22}'K_fA_{22}^{-1}B_{22} \end{bmatrix} \begin{bmatrix} \hat{R}_1 & \hat{Q}_3 \\ \hat{Q}_3' & \hat{R}_2 \end{bmatrix}^{-1} \cdot$$

$$= \left[\begin{array}{c|c} I - B_{21}'(A_{22}^{-1})'K_fB_{21}R_1^{-1} & -B_{21}'(A_{22}^{-1})'K_fB_{22}R_2^{-1} \\ \hline -B_{22}'(A_{22}^{-1})'K_fB_{21}R_1^{-1} & I - B_{22}'(A_{22}^{-1})'K_fB_{22}R_2^{-1} \end{array}\right]^{-1} \tag{C.4}$$

which, when substituted into equation (2.24) with $S_1 = K_s$ and $K_f = S_3$ shows that

$$u_{ic}(t) = u_i(t) + O(\mu). \tag{C.5}$$

APPENDIX D

PROOF OF THEOREM 2.3

If P_c and S possess power series expansions about $\mu = 0$ it is easy to show that their difference also possesses a power series expansion about $\mu = 0$. Subtracting (2.4) from (2.33) we get a new Lyapunov equation in $W = P_c - S$.

$$0 = W[A-\tilde{B}M_c] + [A-\tilde{B}M_c]'W + [S-M_c']\tilde{B}[S-M_c] \tag{D.1}$$

where

$$\tilde{B} = B_1R_1^{-1}B_1' + B_2R_2^{-1}B_2'.$$

Substituting the power series expansion for S it is seen that

$$(S-M_c')\tilde{B}(S-M_c) = 0(\mu^2). \tag{D.2}$$

Expanding W in a power series about $\mu = 0$

$$W = \sum_{i=0}^{\infty} \frac{\mu^i}{i!} \begin{bmatrix} W_1^{(i)} & \mu W_2^{(i)} \\ \mu W_2'^{(i)} & \mu W_3^{(i)} \end{bmatrix}. \tag{D.3}$$

Substituting (D.3) into (D.1) it can be seen that $W = 0(\mu^2)$ and hence we have that

$$J_c = J_{opt} + 0(\mu^2). \tag{D.4}$$

APPENDIX E

PROOF OF THEOREM 2.4

From the Nash equilibrium strategy definition and the relationship between the composite cost (2.32) and the optimal Nash cost (2.34) we have the relationship

$$J_1(u_{1c}, u_{2c}) \leq J_1(u_1, u_2^*) + 0(\mu^2) \tag{E.1a}$$

$$J_2(u_{1c}, u_{2c}) \leq J_2(u_1^*, u_2) + 0(\mu^2) \tag{E.1b}$$

where u_{1c} is the composite control (2.30) for player i, u_i^* the optimal Nash control (2.5), and u_i is some other control in an appropriate set. If we apply

$$u_1 = F_{11}x_1 + F_{12}x_2 \tag{E.2a}$$

$$u_2^* = F_{21}^*x_1 + F_{22}^*x_2 \tag{E.2b}$$

to (2.1) for cost (2.2) we get a performance cost which can be expressed as

$$J_1(u_1, u_2^*) = \frac{1}{2} x_0'P_1x_0 . \tag{E.3}$$

If the set (u_1, u_{2c}) where

$$u_{2c} = [F_{21}^* + 0(\mu)]x_1 + [F_{22}^* + 0(\mu)]x_2 \tag{E.4}$$

is applied to (2.1) for cost (2.2) we get a cost

$$J_1(u_1, u_{2c}) = \frac{1}{2} x_o'C_1x_1 . \tag{E.5}$$

We assume that P_1 and C_1 possess power series expansions about $\mu = 0$. Subtracting the Lyapunov equations for P_1 and C_1 we get a Lyapunov equation in $Y_1 = C_1 - P_1$ given by

$$0 = Y_1\{A + B_1[F_{11} \vdots F_{12}] + B_2[F_{21}^* \vdots F_{22}^*]\} + \{A + B_1[F_{11} \vdots F_{12}] + B_2[F_{21}^* \vdots F_{22}^*]\}' Y_1$$

$$+ C_1 B_2[0(\mu)] + [0(\mu)]' B_2' C_1 + 0(\mu^2). \tag{E.6}$$

Since C is typically of the form

$$C_1 = \begin{bmatrix} C_{11}(\mu) & \mu C_{12}(\mu) \\ \mu C_{12}'(\mu) & \mu C_{13}(\mu) \end{bmatrix} \tag{E.7}$$

we see that

$$C_1 = P_1 + 0(\mu) \tag{E.8}$$

and hence at $\mu = 0$ $J_1(\mu_1, u_2^*) = J_1(u_1, u_{2c})$. Thus we can substitute $J_1(u_1, u_{2c})$ into (E.1a) to give

$$J_1(u_{1c}, u_{2c}) \leq J_1(u_1, u_{2c}) + 0(\mu). \tag{E.9}$$

We can perform similar computations for J_2 and combine the result with (E.9) to give (2.36).

APPENDIX F

PROOF OF THEOREM 3.1

The approach to the proof of Theorem 3.1 is to represent $\overline{K}_i$ as given by (3.27). When this form is substituted into (3.22) we will show that, under the conditions of Theorem 3.1, each term in the series expansion of $\overline{K}_i$ about $\mu = 0$ exists and is unique. Then, clearly, there is a $\mu^* > 0$ small enough to guarantee convergence of the series for all $0 < \mu \leq \mu^*$.

The substitution of (3.27) into (3.22) at $\mu = 0$ yields equations (3.28)-(3.31); $i,j = 1,2$; $i \neq j$. If

$$\overline{K}_{i3}^{(0)} = 0 \qquad , \; i = 1,2 \tag{F.1}$$

is the unique positive semidefinite solution to (3.30), then (F.1) may be substituted into (3.29) to uniquely yield

$$\overline{K}_{i2}^{(0)} = -\overline{K}_{i1}^{(0)} A_{12} A_{22}^{-1}. \tag{F.2}$$

Substitution of (F.2) and (F.1) into (3.28) and manipulating gives

$$\begin{aligned} 0 = \hat{Q}_{i1} + \overline{K}_{i1}^{(0)} A_0 + A_0' \overline{K}_{i1}^{(0)} - [\overline{K}_{i1}^{(0)} B_{0j} + \hat{Q}_{i2} B_{2j}] \tilde{M}_j - \tilde{M}_j' [B_{0j}' \overline{K}_{i1}^{(0)} + B_{2j}' \hat{Q}_{i2}'] \\ + \tilde{M}_j' \hat{R}_{ij} \tilde{M}_j - \tilde{M}_i' \hat{R}_{ii} \tilde{M}_i \end{aligned} \tag{F.3}$$

where

$$\tilde{M}_i = [\hat{R}_{ii} - \hat{Q}_{i3} \hat{R}_{jj}^{-1} \hat{Q}_{j3}]^{-1} [B_{2j}' \hat{Q}_{i2}' + B_{0i}' \overline{K}_{i1}^{(0)} + \hat{Q}_{i3} \hat{R}_{jj}^{-1} (B_{2j}' \hat{Q}_{j2}' - B_{0j}' \overline{K}_{j1}^{(0)})]$$

$$i,j = 1,2; \quad i \neq j. \tag{F.4}$$

Comparison of (F.3) and (3.9) show that the two equations are identical with K_{is} appearing in (3.9) where $\overline{K}_{i1}^{(0)}$ appears in (F.3). Thus, if K_{is}, $i = 1,2$, is the unique stabilizing solution to (3.9)

$$\overline{K}_{i1}^{(0)} = K_{is} \qquad , \; i = 1,2. \tag{F.5}$$

Substitution of (F.5) into (F.2) gives

$$\bar{K}_{i2}^{(0)} = -K_{is}A_{12}A_{22}^{-1} \quad , \; i=1,2. \tag{F.6}$$

Thus, we have shown that the first term of the series exists and is unique.

To see if the second term of the series exists we substitute (3.27) into (3.22) and take the first partial with respect to μ at $\mu = 0$. This gives, with some manipulation,

$$0 = \bar{K}_{i3}^{(1)}A_{22} + A_{22}'\bar{K}_{i3}^{(1)} - [(A_{12}A_{22}^{-1})'\bar{K}_{i1}^{(0)}A_{12} + A_{12}'\bar{K}_{i1}^{(0)}A_{12}A_{22}^{-1}] \quad , \; i=1,2. \tag{F.7}$$

If A_{22} is stable (F.7) possesses a unique solution. If we now assume that $\bar{K}_{i3}^{(1)}$ is known from (F.7), $\bar{K}_{i2}^{(1)}$ can be found to be

$$\bar{K}_{i2}^{(1)} = -\bar{K}_{i1}^{(1)}A_{12}A_{22}^{-1} + \mathfrak{R}_i A_{22}^{-1}, \qquad i=1,2 \tag{F.8}$$

where $\mathfrak{R}_i$ is some known matrix.

Substitution of (F.8) into the equation for $\bar{K}_{i1}^{(1)}$ gives

$$\bar{K}_{i1}^{(1)}\hat{A}_i + \hat{A}_i'\bar{K}_{i1}^{(1)} + \bar{K}_{j1}^{(1)}\hat{B}_i + \hat{B}_i'\bar{K}_{j1}^{(1)} = \bar{\mathfrak{R}}_i \quad , \; i,j=1,2, \; i \neq j \tag{F.9}$$

where

$$\hat{A}_i = A_0 - B_{0j}M_{js} + B_{0i}\hat{R}_{ii}^{-1}\hat{Q}_{j3}'\tilde{R}_{jj}^{-1}[B_{2j}'\hat{Q}_{i2}' + B_{0j}'K_{is} - \hat{R}_{ij}M_{js}] - B_{0i}\tilde{R}_{ii}^{-1}\hat{R}_{ii}M_{is} \tag{F.10}$$

$$\hat{B}_i = -B_{0j}\tilde{R}_{jj}^{-1}[B_{2j}'\hat{Q}_{i2}' + B_{0j}'K_{is} - \hat{R}_{ij}M_{js}] + B_{0j}\hat{R}_{jj}^{-1}\hat{Q}_{i3}'\tilde{R}_{ii}^{-1}\hat{R}_{ii}M_{is} \tag{F.11}$$

$$\tilde{R}_{ii} = \hat{R}_{ii} - \hat{Q}_{i3}\hat{R}_{jj}^{-1}\hat{Q}_{j3}. \tag{F.12}$$

To find conditions for $\bar{K}_{11}^{(1)}$ and $\bar{K}_{21}^{(1)}$ to exist and be unique we apply the Kronecker product operator to (F.9) to give the vector form

$$\mathcal{A}_1 k_{11} + \mathcal{B}_1 k_{21} = r_1 \tag{F.13}$$

$$\mathcal{A}_2 k_{21} + \mathcal{B}_2 k_{11} = r_2. \tag{F.14}$$

Then if

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_1 \end{bmatrix} \tag{F.15}$$

is nonsingular $\bar{K}_{11}^{(1)}$ and $\bar{K}_{21}^{(1)}$ exist and are unique.

The existence of higher derivatives follows in an analogous manner and existence and uniqueness are guaranteed by A_{22} and (F.15) nonsingular. Instead of giving a specific manner in which a unique solution exists to (F.9) we could just specify that if there exists a unique solution to (F.9) then the power series exists and is unique.

APPENDIX G

PROOF OF THEOREM 3.2

Represent P_{ir} as

$$P_{ir} = \begin{bmatrix} P_{i1}(\mu) & \mu P_{i2}(\mu) \\ \mu P_{i2}'(\mu) & \mu P_{i3}(\mu) \end{bmatrix}. \tag{G.1}$$

Then denote $P_{ik}^{(j)} = \left.\dfrac{\partial^j P_{ik}(\mu)}{\partial \mu^j}\right|_{\mu=0}$, $i = 1,2;\ j = 1,2,\ldots;\ k = 1,2,3.$
Substitution of (G.1) into (3.26) at $\mu = 0$ gives

$$\begin{aligned} 0 = {} & P_{i1}^{(0)}[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}] + P_{i2}^{(0)}[A_{21}-B_{2i}M_{is}-B_{2j}M_{js}] \\ & +[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}]'P_{i1}^{(0)} + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'P_{i2}^{(0)\prime} + \xi_i \end{aligned} \tag{G.2a}$$

$$0 = P_{i1}^{(0)}A_{12} + P_{i2}^{(0)}A_{22} + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'P_{i3}^{(0)} \tag{G.2b}$$

$$0 = P_{i3}^{(0)}A_{22} + A_{22}'P_{i3}^{(0)}. \tag{G.2c}$$

If A_{22} is stable then

$$P_{i3}^{(0)} = 0 \tag{G.3}$$

is the unique solution to (G.2c). Substitution of this into (G.2b) gives

$$P_{i2}^{(0)} = -P_{i1}^{(0)}A_{12}A_{22}^{-1}. \tag{G.4}$$

Finally, substitution of (G.4) into (G.2a) gives

$$0 = P_{i1}^{(0)}[A_0-B_{0i}M_{is}-B_{0j}M_{js}] + [A_0-B_{0i}M_{is}-B_{0j}M_{js}]'P_{i1}^{(0)} + \xi \quad ,i,j = 1,2;\ i \neq j. \tag{G.5}$$

Since $[A_0-B_{0i}M_{is}-B_{0j}M_{js}]$ is stable if the slow game possesses a unique stabilizing pair K_{1s},K_{2s}, (G.5) has a unique solution. Hence the first term in (G.1) exists and is unique.

We next examine the existence and uniqueness of the second term in (G.1). Substitute (G.1) into (3.26) and take the partial with respect to μ at $\mu = 0$. This gives

$$0 = P_{i1}^{(1)}[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}] + P_{i2}^{(1)}[A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]$$
$$+[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}]'P_{i1}^{(1)} + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'P_{i2}^{(0)'} \qquad \text{(G.6a)}$$

$$0 = P_{i1}^{(1)}A_{12} + P_{i2}^{(1)}A_{22} + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'P_{i3}^{(1)}$$
$$+[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}]'P_{i2}^{(0)} \qquad \text{(G.6b)}$$

$$0 = P_{i3}^{(1)}A_{22} + A_{22}'P_{i3}^{(1)} + [P_{i2}'^{(0)}A_{12} + A_{12}'P_{i2}^{(0)}] \quad , \; i,j - 1,2; \;\; i \neq j. \qquad \text{(G.6c)}$$

Since $P_{i2}^{(0)}$ are known from the calculations for the first term in the expansion, if A_{22} is stable (G.6c) possesses a unique solution. Then $P_{i2}^{(1)}$ may be found as

$$P_{i2}^{(1)} = -P_{i1}^{(1)}A_{12}A_{22}^{-1} - \mathscr{S}_i A_{22}^{-1} \quad , \; i = 1,2 \qquad \text{(G.7)}$$

where $\mathscr{S}_i$ is a known matrix. Substitution of (G.7) into (G.6a) gives

$$0 = P_{i1}^{(1)}[A_0-B_{0i}M_{is}-B_{0j}M_{js}] + [A_0-B_{0i}M_{is}-B_{0j}M_{js}]'P_{i1}^{(1)} + \mathcal{U}_i \qquad \text{(G.8)}$$

where $\mathcal{U}_i$ is a known matrix. If $[A_0-B_{0i}M_{is}-B_{0j}M_{js}]$ is stable then (G.8) possesses a unique solution. Hence the second term in the series exists and is unique. Higher order terms follow easily and have the same requirements for existence and uniqueness. Since each term of the series exists, clearly, there is a $\bar{\mu}^* > 0$ small enough to guarantee convergence of the series for all $0 < \mu \leq \bar{\mu}^*$.

APPENDIX H

PROOF OF THEOREM 3.3

Subtracting (3.22) from (3.26), we obtain a Lyapunov equation for $\overline{W}_i = P_{ir} - \overline{K}_i$

$$\overline{W}_i \left[\begin{array}{c|c} A_{11}-B_{1i}M_{is}-B_{1j}M_{js} & A_{12} \\ \hline \frac{1}{\mu}[A_{21}-B_{2i}M_{is}-B_{2j}M_{js}] & \frac{A_{22}}{\mu} \end{array}\right] + \left[\begin{array}{c|c} A_{11}-B_{1i}M_{is}-B_{1j}M_{js} & A_{12} \\ \hline \frac{1}{\mu}[A_{21}-B_{2i}M_{is}-B_{2j}M_{js}] & \frac{A_{22}}{\mu} \end{array}\right]' \overline{W}_i$$

$$+ \left[\begin{array}{c|c} \xi_i-\hat{Q}_{i1} & 0 \\ \hline 0 & 0 \end{array}\right] - \overline{K}_i \left[\begin{array}{c|c} B_{1i}M_{is}+B_{1j}M_{js} & 0 \\ \hline \frac{1}{\mu}[B_{2i}M_{is}+B_{2j}M_{js}] & 0 \end{array}\right] - \left[\begin{array}{c|c} B_{1i}M_{is}+B_{1j}M_{js} & 0 \\ \hline \frac{1}{\mu}[B_{2i}M_{is}+B_{2j}M_{js}] & 0 \end{array}\right]' \overline{K}_i$$

$$+ \left\{ \overline{K}_i B_j + \left[\begin{array}{c} \hat{Q}_{i2}B_{2j} \\ \hline 0 \end{array}\right] \right\} \overline{M}_j + \overline{M}_j' \left\{ \overline{K}_i B_j + \left[\begin{array}{c} \hat{Q}_{i2}B_{2j} \\ \hline 0 \end{array}\right] \right\}' - \overline{M}_j'\hat{R}_{ij}\overline{M}_j + \overline{M}_i'\hat{R}_{ii}\overline{M}_i = 0. \tag{H.1}$$

$\overline{W}_i$ possesses a power series expansion about $\mu = 0$ since P_{ir} and $\overline{K}_i$ possess a power series at $\mu = 0$. $\overline{W}_i$ can be expanded as

$$\overline{W}_i = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \begin{bmatrix} \overline{W}_{i1}^{(j)} & \mu\overline{W}_{i2}^{(j)} \\ \mu\overline{W}_{i2}'^{(j)} & \mu\overline{W}_{i3}^{(j)} \end{bmatrix} \quad , \ i = 1,2. \tag{H.2}$$

If $\overline{W}_i$ and the power series expansion for $\overline{K}_i$ are substituted into (H.1) we get at $\mu = 0$

$$\overline{W}_{i1}^{(0)}[A_{11}-B_{1i}M_{is}-B_{1j}M_{js}] + [A_{11}-B_{1i}M_{is}-B_{1j}M_{js}]'\overline{W}_{i1}^{(0)}$$
$$+\overline{W}_{i2}^{(0)}[A_{21}-B_{2i}M_{is}-B_{2j}M_{js}] + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'\overline{W}_{i2}'^{(0)} = 0 \tag{H.3}$$

$$\overline{W}_{i3}^{(0)}A_{22} + A_{22}'\overline{W}_{i3}^{(0)} = 0 \tag{H.4}$$

and

$$\overline{W}_{i1}^{(0)}A_{12} + \overline{W}_{i2}^{(0)}A_{22} + [A_{21}-B_{2i}M_{is}-B_{2j}M_{js}]'\overline{W}_{i3}^{(0)} = 0. \tag{H.5}$$

Since A_{22} is stable (H.4) implies that

$$\overline{W}_{i3}^{(0)} = 0. \tag{H.6}$$

Substitution of (H.6) into (H.5) gives

$$\overline{W}_{i2}^{(0)} = -\overline{W}_{i1}^{(0)} A_{12} A_{22}^{-1}. \tag{H.7}$$

Finally, substitution of (H.7) into (H.3) gives

$$0 = \overline{W}_{i1}^{(0)} [A_0 - B_{01} M_{1s} - B_{02} M_{2s}] + [A_0 - B_{01} M_{1s} - B_{02} M_{2s}]' \overline{W}_{i1}^{(0)}. \tag{H.8}$$

The matrix $[A_0 - B_{01} M_{1s} - B_{02} M_{2s}]$ is the feedback matrix of the slow subsystem (3.6) which is stable. Hence

$$\overline{W}_{i1}^{(0)} = 0 \qquad , \; i = 1,2, \tag{H.9}$$

which implies that

$$\overline{W}_{i2}^{(0)} = 0 \qquad , \; i = 1,2. \tag{H.10}$$

Thus we have proven Theorem 3.3.

APPENDIX I

PROOF OF THEOREM 3.5

The proof of Theorem 3.5 is analogous to the proof of Theorem 3.1 however, due to the complexity of some of the manipulations inclusion is warranted. We represente K_i by the form in equation (3.59). We will show that when this form is substituted into (3.4), under the conditions of Theorem 3.5, each term in the series expansion exists and is unique. Then there exists a $\mu^* > 0$ small enough to guarantee convergence of the series for all $0 < \mu \leq \mu^*$. It should be emphasized that this μ^* is not necessarily the same as the μ^* in the proof of the Theorem 3.1.

Substitution of (3.59) into (3.4) at $\mu = 0$ gives

$$
\begin{aligned}
0 = & -[Q_{i1} + K_{i1}^{(0)} A_{11} + A_{11}' K_{i1}^{(0)} + K_{i2}^{(0)} A_{21} + A_{21}' K_{i2}'^{(0)}] \\
& + [K_{i1}^{(0)} B_{1i} + K_{i2}^{(0)} B_{2i}] R_{ii}^{-1} [B_{1i}' K_{i1}^{(0)} + B_{2i}' K_{i2}'^{(0)}] \\
& + [K_{i1}^{(0)} B_{1j} + K_{i2}^{(0)} B_{2j}] R_{jj}^{-1} [B_{1j}' K_{j1}^{(0)} + B_{2j}' K_{j2}'^{(0)}] \\
& + [K_{j1}^{(0)} B_{1j} + K_{j2}^{(0)} B_{2j}] R_{jj}^{-1} [B_{1j}' K_{i1}^{(0)} + B_{2j}' K_{i2}'^{(0)}] \\
& - [K_{j1}^{(0)} B_{1j} + K_{j2}^{(0)} B_{2j}] R_{jj}^{-1} R_{ij} R_{jj}^{-1} [B_{1j}' K_{j1}^{(0)} + B_{2j}' K_{j2}'^{(0)}]
\end{aligned} \tag{I.1}
$$

$$
\begin{aligned}
0 = & -Q_{i2} - K_{i1}^{(0)} A_{12} - K_{i2}^{(0)} A_{22} - A_{21}' K_{i3}^{(0)} + [K_{i1}^{(0)} B_{1i} + K_{i2}^{(0)} B_{2i}] R_{ii}^{-1} B_{2i} K_{i3}^{(0)} \\
& + [K_{i1}^{(0)} B_{1j} + K_{i2}^{(0)} B_{2j}] R_{jj}^{-1} B_{2j}' K_{j3}^{(0)} + [K_{j1}^{(0)} B_{1j} + K_{j2}^{(0)} B_{2j}] R_{jj}^{-1} B_{2j}' K_{i3}^{(0)} \\
& - [K_{j1}^{(0)} B_{1j} + K_{j2}^{(0)} B_{2j}] R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{2j}' K_{j3}^{(0)}
\end{aligned} \tag{I.2}
$$

$$
\begin{aligned}
0 = & -Q_{i3} - K_{i3}^{(0)} A_{22} - A_{22}' K_{i3}^{(0)} + K_{i3}^{(0)} B_{2i} R_{ii}^{-1} B_{2i}' K_{i3}^{(0)} + K_{i3}^{(0)} B_{2j} R_{jj}^{-1} B_{2j}' K_{j3}^{(0)} \\
& + K_{j3}^{(0)} B_{2j} R_{jj}^{-1} B_{2j}' K_{i3}^{(0)} - K_{j3}^{(0)} B_{2j} R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{2j}' K_{j3}^{(0)} \qquad i,j = 1,2 \quad i \neq j.
\end{aligned} \tag{I.3}
$$

If the fast game (3.41a), (3.41b) possesses a unique stabilizing solution then (I.3) possesses a unique solution and

$$K_{i3}^{(0)} = K_{if}. \tag{I.4}$$

Let

$$N_i = K_{i1}^{(0)} B_{1i} + K_{i2}^{(0)} B_{2i}. \tag{I.5}$$

Then (I.2) can be written as, if $\hat{A}_{22}$ is stable,

$$K_{i2}^{(0)} = \{-Q_{i2} - A_{21}' K_{if} - K_{i1}^{(0)} \hat{A}_{12} + N_j [R_{jj}^{-1} B_{2j}' K_{if} - R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_{2j}' K_{jf}]\} \hat{A}_{22}^{-1}. \tag{I.6}$$

We rewrite (I.1) as

$$\begin{aligned} 0 = {} & -Q_{i1} - A_{11}' K_{i1}^{(0)} - K_{i1}^{(0)} A_{11} - A_{21}' K_{i2}'^{(0)} - K_{i2} A_{21} + N_i R_{ii}^{-1} N_i' - N_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} N_j' \\ & + N_j R_{jj}^{-1} [B_{1j}' K_{i1}^{(0)} + B_{2j}' K_{i2}'^{(0)}] + [K_{i1}^{(0)} B_{1j} + K_{i2}^{(0)} B_{2j}] R_{jj}^{-1} N_j'. \end{aligned} \tag{I.7}$$

Substitution of (I.6) for $K_{i2}^{(0)}$ and making use of equation (I.3) to eliminate terms we get

$$\begin{aligned} 0 = {} & -\tilde{Q}_{i1} - K_{i1}^{(0)} \tilde{A}_0 - \tilde{A}_0' K_{i1}^{(0)} + N_i R_{ii}^{-1} N_i' - N_j R_{jj}^{-1} \tilde{R}_{ij} R_{jj}^{-1} N_j' + N_j R_{jj}^{-1} (\tilde{B}_{0j}' K_{i1}^{(0)} + \bar{Q}_{i2}') \\ & + (K_{i1}^{(0)} \tilde{B}_{0j} + \bar{Q}_{i2}) R_{jj}^{-1} N_j'. \end{aligned} \tag{I.8}$$

We next examine N_i. Substituting (I.6) for $K_{i2}^{(0)}$ and noting

$$\tilde{Q}_{i2} = -(Q_{i2} + A_{21}' K_{if}) \hat{A}_{22}^{-1} B_{2i} \tag{I.9}$$

we get

$$N_i = K_{i1}^{(0)} \tilde{B}_{0i} + \tilde{Q}_{i2} - N_j R_{jj}^{-1} \bar{Q}_{i3}'. \tag{I.10}$$

Observing that $\bar{M}_{is}$ from (3.51) is the same as $R_{ii}^{-1} N_i'$ with $K_{i1}^{(0)}$ replacing K_{ism}, it can be seen that if the modified slow game (3.48), (3.49) possesses a unique stabilizing solution then (3.52) and (I.8) are identical and

$$K_{i1}^{(0)} = K_{ism}. \tag{I.11}$$

Substituting for $K_{i1}^{(0)}$ and N_j in (I.6) it can be seen that

$$K_{i2}^{(0)} = K_{im} \tag{I.12}$$

where K_{im} is from equation (3.54). We have thus shown that if the fast and modified slow games possess unique stabilizing solutions, then the first terms in the series expansion of K_i exist and are unique.

Taking the first partial of (3.4) with respect to μ at $\mu=0$ gives, with some manipulation,

$$\begin{aligned} 0 = &-K_{i3}^{(1)}\hat{A}_{22}-\hat{A}_{22}'K_{i3}^{(1)} +[K_{j3}^{(1)}B_{2j}+K_{j2}'^{(0)}B_{1j}][R_{jj}^{-1}B_{2j}'K_{i3}^{(0)}-R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{j3}^{(0)}] \\ &+[R_{jj}^{-1}B_{2j}'K_{i3}^{(0)}-R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{j3}^{(0)}]'[B_{2j}'K_{j3}^{(1)}+B_{1j}'K_{j2}^{(0)}] \\ &-K_{i2}'^{(0)}\hat{A}_{12}-\hat{A}_{12}'K_{i2}^{(0)}. \end{aligned} \tag{I.13}$$

Equation (I.13) can be written as

$$K_{13}^{(1)}\hat{A}_{22} +\hat{A}_{22}'K_{13}^{(1)}-K_{23}^{(1)}C_2-C_2'K_{23}^{(1)} = R_1 \tag{I.14a}$$

$$K_{23}^{(1)}\hat{A}_{22} +\hat{A}_{22}'K_{23}^{(1)}-K_{13}^{(1)}C_1-C_1'K_{13}^{(1)} = R_2 \tag{I.14b}$$

where R_1 and R_2 are some known matrices and

$$C_i = B_{2i}R_{ii}^{-1}B_{2i}'K_{jf}-B_{2i}R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}. \tag{I.15}$$

We can rewrite (I.14) using the Kronecker product operator as

$$\begin{aligned} \mathcal{a}_3 k_{13}^{(1)} - \mathcal{C}_2 k_{23}^{(1)} &= \bar{r}_1 \\ \mathcal{a}_3 k_{23}^{(1)} - \mathcal{C}_1 k_{13}^{(1)} &= \bar{r}_2 \end{aligned} \tag{I.16}$$

where

$$\mathcal{a}_3 = I\otimes \hat{A}_{22}' +\hat{A}_{22}'\otimes I$$

$$\mathcal{C}_i = I\otimes C_i' +C_i'\otimes I.$$

If $\hat{A}_{22}$ is stable then $\mathcal{A}_3$ is stable and $k_{13}^{(1)}$ and $k_{23}^{(1)}$ exist and are unique if $\mathcal{A}_3 - C_1\mathcal{A}_3^{-1}C_2$ is nonsingular. Thus if $\hat{A}_{22}$ is stable and $\mathcal{A}_3 - C_1\mathcal{A}_3^{-1}C_2$ is nonsingular $K_{i3}^{(1)}$ exist and are unique. Assuming that $K_{i3}^{(1)}$ are known we get

$$K_{j2}^{(1)} = K_{i1}^{(1)}N_{i1} + K_{j1}^{(1)}N_{i2} + \mathcal{P}_i \tag{I.17}$$

where $\mathcal{P}_i$ is a known matrix,

$$N_{i1} = \tilde{B}_{01}[R_{ii}^{-1}B_{2i}'K_{jf} - R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}T_i^{-1}$$

$$N_{i2} = \{-\hat{A}_{12}\hat{A}_{22}^{-1} + B_{1j}[R_{jj}^{-1}B_{2j}K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i}[R_{ii}^{-1}B_{2i}'K_{jf}$$
$$- R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}\}T_i^{-1}$$

$$T_i = I - B_{2j}[R_{jj}^{-1}B_{2j}'K_{if} - R_{jj}^{-1}R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}B_{2i}[R_{ii}^{-1}B_{2i}'K_{jf}$$
$$- R_{ii}^{-1}R_{ji}R_{ii}^{-1}B_{2i}'K_{if}]\hat{A}_{22}^{-1}.$$

The equations for $K_{i1}^{(1)}$ are given by

$$0 = -K_{i1}^{(1)}[A_{11} - B_{1i}\bar{M}_{is} - B_{1j}\bar{M}_{js}] - [A_{11} - B_{1i}\bar{M}_{is} - B_{ij}\bar{M}_{js}]'K_{i1}^{(1)}$$
$$+ [K_{j1}^{(0)}B_{1j} + K_{j2}^{(1)}B_{2j}]R_{jj}^{-1}[B_{1j}'K_{i1}^{(0)} + B_{2j}'K_{i2}'^{(0)} - R_{ij}\bar{M}_{js}] + [B_{1j}'K_{i1}^{(0)}$$
$$+ B_{2j}'K_{i2}'^{(0)} - R_{ij}\bar{M}_{js}]'R_{jj}^{-1}[B_{1j}'K_{j1}^{(1)} + B_{2j}'K_{j2}'^{(1)}] + K_{i2}^{(1)}[-A_{21} + B_{2i}\bar{M}_{is}$$
$$+ B_{2j}\bar{M}_{js}] + [-A_{21} + B_{2i}\bar{M}_{is} + B_{2j}\bar{M}_{js}]'K_{i2}'^{(1)}. \tag{I.18}$$

Substitution of (I.17) into (I.18) gives

$$\mathfrak{F}_i = K_{i1}^{(1)}A_i + A_i'K_{i1}^{(1)} - B_i'K_{j1}^{(1)} - K_{j1}^{(1)}B_i \tag{I.19}$$

where $\mathfrak{F}_i$ = known matrix

$$\check{A}_i = \tilde{A}_0 - \tilde{B}_{0i}\bar{M}_{is} - \tilde{B}_{0j}\bar{M}_{js} + N_{i1}B_{2j}R_{jj}^{-1}\tau_i$$

$$B_i = [B_{1j} + N_{i2}B_{2j}]R_{jj}^{-1}\tau_i$$

$$\tau_i = B_{1j}'K_{ism} + B_{2j}'K_{im}' - R_{ij}\overline{M}_{js} + [B_{2j}'K_{if} - R_{ij}R_{jj}^{-1}B_{2j}'K_{jf}]\hat{A}_{22}^{-1}\cdot$$

$$\cdot[-A_{21} + B_{2i}\overline{M}_{is} + B_{2j}\overline{M}_{js}].$$

(I.19) is a set of coupled Lyapunov equations. To guarantee existence and uniqueness of solutions we first use the Kronecker product operator to stack the equations. The stacked equations are

$$\begin{aligned} \check{\alpha}_1 k_{11}^{(1)} + \check{\beta}_1 k_{21}^{(1)} &= f_1 \\ \check{\beta}_2 k_{11}^{(1)} + \check{\alpha}_2 k_{21}^{(1)} &= f_2. \end{aligned} \qquad \text{(I.20)}$$

Then if

$$\begin{bmatrix} \check{\alpha}_1 & \check{\beta}_1 \\ \check{\beta}_2 & \check{\alpha}_2 \end{bmatrix}$$

is nonsingular, the coupled Lyapunov equations have a unique solution.

Thus we have shown that the first two terms of the series for K_i exists and is unique under the conditions given. Higher order terms have the same conditions for existence and uniqueness as the second term.

APPENDIX J

PROOF OF THEOREM 3.10

When the composite control (3.72) is applied to the full order system (3.1) the resulting feedback system is

$$\dot{x}_1 = [A_{11}-B_{11}\bar{M}_{1s}-B_{12}\bar{M}_{2s}]x_1 + [A_{12}-B_{11}R_{11}^{-1}B_{21}'K_{1f}-B_{12}R_{11}^{-1}B_{21}'K_{2f}]x_2 \tag{J.1a}$$

$$\mu\dot{x}_2 = [A_{21}-B_{21}\bar{M}_{1s}-B_{22}\bar{M}_{2s}]x_1 + [A_{22}-B_{21}R_{11}^{-1}B_{21}'K_{1f}-B_{22}R_{22}^{-1}B_{22}'K_{2f}]x_2 \tag{J.1b}$$

or by defining terms

$$\dot{x}_1 = \hat{A}_{11}x_1 + \hat{A}_{12}x_2 \tag{J.2a}$$

$$\mu\dot{x}_2 = \hat{A}_{21}x_1 + \hat{A}_{22}x_2 \tag{J.2b}$$

where $\hat{A}_{12}$ and $\hat{A}_{22}$ are as defined after equation (3.46). Following [22] we construct a transformation

$$T = \begin{bmatrix} I_1-\mu H_1L_1 & -\mu H_1 \\ L_1 & I_2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I_1 & \mu H_1 \\ -L_1 & I_2-\mu L_1H_1 \end{bmatrix} \tag{J.3}$$

to diagonalize (J.2). I_1 and I_2 are $n_1 \times n_1$ and $n_2 \times n_2$ dimension identity matrices and H_1 and L_1 will be defined in the following manipulations. Let

$$\eta_1 = x_2 + \hat{A}_{22}^{-1}\hat{A}_{21}x_1 + \mu N_1x_1 \tag{J.4a}$$

$$= x_2 + L_1x_1 \tag{J.4b}$$

and hence

$$L_1 = \hat{A}_{22}^{-1}\hat{A}_{21}x_1 + \mu N_1 x_1. \tag{J.5}$$

Substituting (J.4b) into (J.2) we get

$$\dot{x}_1 = [\hat{A}_{11} - \hat{A}_{12}L_1]x_1 + \hat{A}_{12}\eta_1 \tag{J.6a}$$

$$\mu\dot{\eta}_1 = [\hat{A}_{21} - \hat{A}_{22}L_1 + \mu L_1\hat{A}_{11} - \mu L_1\hat{A}_{12}L_1]x_1 + [\hat{A}_{22} + \mu L_1\hat{A}_{12}]\eta_1. \tag{J.6b}$$

We wish the block multiplying x_1 in the $\dot{\eta}$ equation to be zero. Substituting for L_1 in this block we get

$$\begin{aligned} 0 = {} & -\hat{A}_{22}N_1 + \hat{A}_{22}^{-1}\hat{A}_{21}\hat{A}_{11} + \mu N_1\hat{A}_{11} - \hat{A}_{21}^{-1}\hat{A}_{21}\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} \\ & -\mu N_1\hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} - \mu\hat{A}_{22}^{-1}\hat{A}_{21}\hat{A}_{12}N_1 - \mu^2 N\hat{A}_{12}N_1. \end{aligned} \tag{J.7}$$

By the Implicit Function Theorem we get

$$N_1 = \hat{A}_{22}^{-1}\hat{A}_{22}^{-1}\hat{A}_{21}[\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}] + O(\mu) \tag{J.8a}$$

$$= \hat{A}_{22}^{-1}\hat{A}_{22}^{-1}\hat{A}_{21}[\tilde{A}_0 - \tilde{B}_{01}\overline{M}_{1s} - \tilde{B}_{02}\overline{M}_{2s}] + O(\mu). \tag{J.8b}$$

To $O(\mu)$ the system is now upper block triangular. To complete the diagonalization we let

$$\xi_1 = x_1 - \mu H\eta_1. \tag{J.9}$$

Substituting for x_1 in (J.6a) we get

$$\dot{\xi}_1 = [\hat{A}_{11} - \hat{A}_{12}L_1]\xi_1 + [\hat{A}_{12} + \mu(\hat{A}_{11} - \hat{A}_{12}L_1)H_1 - H_1(\mu L_1\hat{A}_{12} + \hat{A}_{22})]\eta_1. \tag{J.10}$$

We wish the term multiplying η to be zero. By the Implicit Function Theorem this gives

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$$H_1 = \hat{A}_{12}\hat{A}_{22}^{-1} + O(\mu). \tag{J.11}$$

Solving for the state trajectories for ξ_1 and η_1 and then applying the inverse transformation back into x_1 and x_2 coordinates we get

$$x_1(t) = \{\exp([\hat{A}_0 - \tilde{B}_{01}\bar{M}_{1s} - \tilde{B}_{02}\bar{M}_{2s}]t)\}x_s(0) + O(\mu) \tag{J.12a}$$

$$x_2(t) = -\hat{A}_{22}^{-1}\hat{A}_{21}\{\exp([\tilde{A}_0 - \tilde{B}_{01}\bar{M}_{1s} - \tilde{B}_{02}\bar{M}_{2s}]t)\}x_s(0) \tag{J.12b}$$

$$+\{\exp([\hat{A}_{22}]^{t}/\mu)\}x_f(0) + O(\mu).$$

By comparing these equations with the equations resulting when the modified slow control is applied to the modified slow subsystem and when the fast control is applied to the fast subsystem we get

$$x_1(t) = x_{sm}(t) + O(\mu) \tag{J.13a}$$

$$x_2(t) = -\hat{A}_{22}^{-1}\hat{A}_{21}x_{sm}(t) + x_f(t) + O(\mu). \tag{J.13b}$$

If $[\tilde{A}_0 - \tilde{B}_{01}\bar{M}_{1s} - \tilde{B}_{02}\bar{M}_{2s}]$ is stable (J.13a) and (J.13b) hold for all finite $t > 0$. If $\hat{A}_{22}$ is also stable then they hold for all $t \in [0, \infty)$.

VITA

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